SOME RESULTS ABOUT COMPACT ELEMENTS IN C^{+}_{-} ALGEBRAS

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(SIRAJ AHMED ANSARI)

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PREFACE

It is well-known that an operator T on a Banach space E is invertible if and only if $||T-I|| < 1$, where I is the identity operator on E. Replacing the role of the Banach algebra $L(E)$ of bounded linear operators on E in the aforementioned result by an arbitrary Banach algebra with identity, this result finds far reaching generalizations. This change of the $L(E)$-situation to arbitrary Banach algebras suggests the possibilities of extending various notions and results known for the algebra $L(E)$ in the setting of arbitrary Banach algebras. Indeed, a beginning in this direction was already made by M. Freundlich in 1949 in her work [12] on 'Completely continuous elements of a normed ring'. This study was carried further by K. Vala [25] in 1967 and a year later, almost simultaneously, by K. Ylinen [27] and J.C. Alexander [1]. Here they discuss compact elements in certain Banach algebras and in the process prove some significant results in this more general setting. One of the most important results in this direction was proved by K. Ylinen [28] in 1972 which states that the compact (resp. finite-dimensional) elements in an arbitrary C^*-algebra A are precisely (preimages of) the compact (resp. finite-dimensional) operators on some Hilbert space H under a suitable faithful $*$-isometric representation of A over H. On the other hand, the notion of a trace-class operator, which was at the basis of Mathematical foundations of

Quantum Mechanics as envisaged by J. von Neumann, has also been the subject-matter of active research in case of certain Banach algebras. In this connection Puhl [20], while carrying out this line of investigation, proves a striking analogue of the famous Lidskij's theorem on the eigenvalues of nuclear operators in Hilbert spaces. Recently K. Astala and M.S. Ramanujan [3] have defined approximation numbers of elements in Banach algebras and have, in the process, established a remarkable analogue of the celebrated spectral decomposition theorem in the context of C^*-algebras.

The present dissertation makes an attempt to offer a concise and coherent treatment of the notions and results on the above subject-matter scattered in the literature. It consists of four chapters and each chapter comprises various sections, the sections are numbered in the order in which they occur.

In Chapter I, we have described those preliminary concepts which are used throughout the text. This chapter is primarily aimed at making the present dissertation as self-contained as possible.

Chapter II incorporates the notions of completely continuous, compact and weakly completely continuous elements in the setting of Banach algebras and it has four sections. While the first section presents a brief survey of the above mentioned concepts, Section second is devoted to the study of completely continuous elements of normed algebras as introduced by M. Freundlich. In the third section compact elements of normed algebras in the sense of K. Vala have been taken. In the last section of this chapter equivalence of the notions of compactness and weak complete continuity of elements in the case of C^*-algebras have been discussed.

Chapter III comprises three sections, where the notions of single, finite, and nuclear elements have been discussed. The first section being introductory, second section treats single elements and the last section of this chapter deals with finite and nuclear elements in the situation of certain Banach algebras. Analogous to the classical situation in operator theory we define the trace of these elements in the case of some suitable Banach algebras and establish a Lidskij-type result on the trace of elements in C^*-algebras.

The last chapter deals with the problem of characterizing compact elements in a C^*-algebra A as preimages of compact operators on a Hilbert space H via the existence of an $*$-isometric representation of A over H. This type of characterization has also been given in case of finite-dimensional elements.

The dissertation ends with a small bibliography which by no means is exhaustive but contains mostly those references which are referred to in the text.

CHAPTER - I

PRELIMINARIES

1.1 INTRODUCTION : The purpose of this chapter is to introduce basic concepts, preliminary notions and some fundamental results which will serve as a ready reference to the prerequisites for the development of the subject embodied in the present dissertation. Only those notions and results that have been used in the later chapters are presented in order to make the text self contained as far as possible. For most of the material in this chapter, we refer to Dunford and Schwartz [10], Larsen [14], Pietsch ([18], [19]), Rickart [21] and Takesaki [22].

To begin with we have the following :

1.2.1 DEFINITION : An algebra A over a field $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$) is a linear space over $\mathbb{K}$ equipped with an associative and distributive binary operation called multiplication, satisfying the following condition :

$$a(xy) = (ax)y = x(ay), \text{ for every } a \in \mathbb{K} \text{ and } x,y \in A$$

1.2.2 DEFINITION : An algebra A is said to have an identity if there exists an element e in A such that $xe = ex = x$ for all x in A.

A is said to be a commutative algebra if $xy = yx$ for every x, y in A.

1.2.3 <u>DEFINITION</u> : Let A be an algebra such that A is a normed space satisfying the multiplicative inequility

$$||xy|| \leq ||x|| \, ||y||, \text{ for all } x,y \text{ in } A.$$

Then A is said to be a normed algebra.

A complete normed algebra is called a Banach algebra.

1.2.4 <u>DEFINITION</u> : Let A be an algebra. Then $I \subset A$ is said to be a left (right) ideal in A if I is a linear subspace of A such that $xI \subset I (Ix \subset I)$ for all $x \in A$; $I \subset A$ is said to be a two-sided ideal (or simply an ideal) if I is both a left and a right ideal in A. An ideal $I \subset A$ is proper if $I \neq A$, and a proper left (right, two-sided) ideal I is said to be maximal if, whenever $J \subset A$ is a left (right, two-sided) ideal in A such that $I \subset J$, then either $I = J$ or $J = A$. Furthermore, $I \subset A$ is said to be a sub-algebra if I is a linear subspace such that $x,y \in I$ implies $xy \in I$.

Evidently every ideal in an algebra A is a sub-algebra, and in a commutative algebra all ideals are two-sided.

<u>REMARK</u> : An algebra without an identity can be embedded into an algebra with identity.

1.2.5 <u>DEFINITION</u> : Let A be an algebra with identity e. An element $x \in A$ is said to have a left (right) inverse if there exists some $y \in A$ such that $yx = e$ ($xy = e$), whereas x is said to have an inverse if there exists some $y \in A$ such that $xy = yx = e$. If $x \in A$ has an inverse, then x is said to be a unit (regular, or invertible), otherwise it is said to be singular.

1.2.6 <u>DEFINITION</u> : An ideal in an algebra A generated by an element of A is called a principal ideal. The principal ideal generated by the element r in A will be denoted by $\{r\}$.

1.2.7 <u>DEFINITION</u> : Let A be an algebra. A left (right, two-sided) ideal I in A is said to be regular (modular) if there exists some $u \in A$ such that $xu - x \in I$ ($ux - x \in I$, $xu - x \in I$ and $ux - x \in I$), for all x in A.

1.2.8 <u>DEFINITION</u> : A two-sided ideal in an algebra A is called primitive if it is the quotient of a maximal regular left ideal. The algebra A is said to be primitive if the zero ideal is the primitive ideal.

1.2.9 <u>DEFINITION</u> : An ideal I in an algebra A is said to be minimal if it is different from (o) and does not contain properly any ideal other than (o).

Let $\{I_\lambda : \lambda \in \Lambda\}$ be a family of left (right) ideals in an algebra A. Then the smallest left (right) ideal in A which contains every I_λ is called the sum of the ideals

I_λ. In an arbitrary algebra A, the sum of the minimal left (right) ideals is called the left (right) socle of A. If A does not contain minimal left (right) ideals then the socle of A is defined to be equal to (o). If the left socle is equal to the right socle, it is simply called the socle of A.

1.2.10 DUAL ALGEBRAS : Let E be an arbitrary subset of an algebra A and let

$$A_\ell(E) = \{x \in A : xE = (o)\}$$

and

$$A_r(E) = \{x \in A : Ex = (o)\}.$$

Then $A_\ell(E)$ is called the left annihilator and $A_r(E)$ the right annihilator of E. A left ideal L is called a left annihilator ideal if it has the form $L = A_\ell(E)$ for some set $E \subseteq A$. Similarly, a right ideal of the form $A_r(E)$ is called a right annihilator ideal. It is easily seen that an ideal L is a left annihilator ideal if and only if $L = A_\ell(A_r(L))$. Similarly, R is a right annihilator ideal if and only if $R = A_r\,(A_\ell(R))$. Finally, we note that the left (right) annihilator of a left (right) ideal is a two-sided ideal.

If every closed (left or right) ideal in an algebra A is an annihilator ideal, then A is called a dual algebra.

1.2.11 DEFINITION : Let A be a normed algebra. Then $x \in A$

is said to be nilpotent if there exists some non-negative integer n such that $x^n = o$, and $x \in A$ is said to be topologically nilpotent if $\lim_n ||x^n||^{1/n} = o$.

An element e in a normed algebra A is said to be idempotent if $e^2 = e$, and an idempotent e such that eAe is a division algebra is said to be a minimal idempotent. Two idempotents e_1 and e_2 such that $e_1e_2 = e_2e_1 = o$ are said to be orthogonal.

In order to conclude this section with Gel'fand Representation Theorem, we need to state some more definitions and results.

1.2.12 <u>DEFINITION</u> : Let A be a normed algebra. A homomorphism τ of A onto $\mathbb{C}$ is said to be a complex homomorphism (multiplicative linear functional).

Regarding complex homomorphisms and maximal regular ideals in a commutative Banach algebra A, we have the following :

1.2.13 <u>THEOREM</u> [14] Let A be a commutative Banach algebra. Then

(i) If τ is a complex homomorphism of A, then τ is continuous and $||\tau|| \leq 1$. Moreover, if A has an identity e and $||e|| = 1$, then $||\tau|| = 1$.

(ii) If τ is a complex homomorphism of A, then $M = \tau^{-1}(o)$

is a maximal regular ideal in A.

(iii) If $M \subset A$ is a maximal regular ideal, then there exists a unique complex homomorphism τ of A such that $\tau^{-1}(o) = M$.

(iv) The correspondence between complex homomorphisms of A and the maximal regular ideals in A determined by parts (ii) and (iii) is bijective.

1.2.14 <u>DEFINITION</u> : Let A be a commutative Banach algebra, then the collection of all the maximal regular ideals M in A (denoted by $\Delta(A)$) is called the maximal ideal space (structure space) of A.

1.2.15 <u>DEFINITION</u> : Let A be a commutative Banach algebra. The Gel'fand topology on $\Delta(A)$ is defined to be the relative weak*-topology on $\Delta(A)$ considered as a subset of A^*, the space of continuous linear functionals on A, by means of the bijective map $\tau \longrightarrow \tau^{-1}(o)$ in (IV) of Th. 1.2.13.

The salient properties of the ideal space $\Delta(A)$ with the Gel'fand topology are given by the following:

1.2.16 <u>THEOREM</u> : Let A be a commutative Banach algebra. Then,

(i) If A has an identity, then $\Delta(A)$ with the Gel'fand topology is a compact Hausdorff topological space.

(ii) If A is without identity, then $\Delta(A)$ with the Gel'fand topology is a locally compact Hausdorff topological space, and $\Delta(A[e])$ with the Gelfand topology is the one-point compactification of $\Delta(A)$, where $A[e]$ is the Banach algebra obtained by adjoining an identity to A.

In order to represent a commutative Banach algebra A homomorphically as an algebra of continuous functions on the locally compact Hausdorff space $\Delta(A)$, an appropriate mapping from A to the continuous functions on $\Delta(A)$ is defined as follows :

1.2.17 DEFINITION : Let A be a commutative Banach algebra. If $x \in A$, then $\hat{x}$ will denote the complex valued function defined on $\Delta(A)$ by $\hat{x}(M) = \hat{x}[\tau^{-1}(o)] = \tau(x)$, $\tau \in \Delta(A)$. (Here M is again identified with τ by means of IV in Th. 1.2.13).

Following is the Gel'fand Representation Theorem :

1.2.18 THEOREM [14] (Gel'fand Representation Theorem) : Let A be a commutative Banach algebra. The mapping $x \longrightarrow \hat{x}$, $x \in A$, defines a homomorphism of A onto a subalgebra A of $C_o(\Delta(A))$, where $C_o(\Delta(A))$ is the algebra of all continuous complex-valued functions on $\Delta(A)$ that vanish at infinity Moreover, $\hat{A}$ seperates the points of $\Delta(A)$, and if A has an identity, then $\hat{A}$ contains the constant functions.

For a commutative Banach algebra A, the mapping

$x \longrightarrow \hat{x}$, $x \in A$ is said to be the Gel'fand transformation and if the Gel'fand transformation on A is injective, then A is said to be semisimple.

1.2.19 DEFINITION : For a commutative Banach algebra A the radical of A, denoted by Rad (A), is defined as the intersection of all the maximal regular ideals in A; that is,

$$\mathrm{Rad}\ (A) = \bigcap_{M \in \Delta(A)} M$$

If $\Delta(A) = \Phi$ then Rad (A) is identified with A, and in this case A itself is said to be a radical algebra.

1.3.1 DEFINITION : Let A be a Banach algebra. Then A is said to be a Banach algebra with involution if there exists a mapping $* : A \to A$ such that for any $x, y \in A$ and $a \in \mathbb{C}$ we have

(i) $(x+y)^* = x^* + y^*$,

(ii) $(ax)^* = \bar{a}\, x^*$,

(iii) $(xy)^* = y^* x^*$,

(iv) $(x^*)^* = x^{**} = x$.

1.3.2 DEFINITION : Let A and B be Banach algebras with involutions $*$ and $\diamond$, respectively. A homomorphism $f : A \longrightarrow B$ is said to be a *-homomorphism if

$f(x^*) = f(x)^+$, $x \in A$.

A Banach algebra A with involution is said to be a B^*-algebra if $||x^*x|| = ||x||^2$, $x \in A$. A norm-closed subalgebra of L(H) which is closed with respect to the involution $T \longrightarrow T^*$ is called a C^*-algebra.

<u>REMARK</u> : Every C^*-algebra is a B^*-algebra and every complex B^*-algebra is isometrically $*$-isomorphic to a C^*-algebra.

1.3.3 <u>DEFINITION</u> : Let A be a Banach algebra with involution. An element $x \in A$ is said to be self-adjoint (hermitian) if $x^* = x$, and an idempotent p such that $p^* = p$ is called a projection. An element $x \in A$ such that $xx^* = x^*x$ is called normal.

1.3.4 <u>THE REPRESENTATIONS</u> : Let A be any algebra over a field F. Let X be a linear space over the same field F and denote by L(X) the algebra of all linear transformations of X into itself. Then any homomorphism of A into the algebra L(X) is called a representation of A in L(X) or on X. The representation is called faithful if the homomorphism is an isomorphism. If the field is $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and X is a normed linear space, then a homomorphism of A into the algebra B(X) of all bounded linear transformations of X into itself is called a normed representation. For the case of a normed algebra the term representation will mean normed representation. A representation $a \longrightarrow Ta$ of a normed algebra A on X is said to be (uniformly) continuous or bounded provided there exists a constant c such that $||Ta|| \leq c||a||$ for all $a \in A$.

In the sequel we shall be primarily concerned with the representations of Banach algebras with involutions or *-algebras, therefore in the context of such algebras there is again a pressing need to reformulate the above definition of a representation in more explicit terms.

1.3.5 <u>DEFINITION</u> : Let A be a Banach algebra with involution. A representation of A is a *-homomorphism π of A into the C^*-algebra $L(H)$ of all bounded linear operators on a given Hilbert space H. The Hilbert space H is called the representation space of π. Such a representation is explicitly indicated by $\{\pi, H\}$. If $\pi(x) \neq o$ for every non-zero $x \in A$, then π is called faithful.

1.3.6 <u>DEFINITION</u> : Two representations $\{\pi_1, H_1\}$ and $\{\pi_2, H_2\}$ of A are said to be unitarily equivalent if there exists an isometry U of H_1 onto H_2 such that $U \pi_1(x) U^* = \pi_2(x)$, $x \in A$.

Given a representation $\{\pi, H\}$ of A, a closed subspace M of H is called an invarient subspace of $\{\pi, H\}$, if $\pi(x) M \subset M$ for every $x \in A$. If $\{\pi, H\}$ has no invariant subspace other than H and $\{o\}$, then it is said to be irreducible (or, more precisely, topologically irreducible).

In the next section we present those results and concepts that have not been introduced in the preceding definitions.

1.4.1 <u>DEFINITION</u> : Let A be a commutative Banach algebra. A net $\{u_\alpha\} \subset A$ is said to be an approximate identity if,

(i) $\sup_\alpha ||u_\alpha|| < \infty$,

(ii) $\lim_\alpha ||u_\alpha x - x|| = o$, $x \in A$.

<u>NOTATION</u> : $J(E)$ will denote the set of all operators of finite rank on a normed space E.

1.4.2 <u>DEFINITION</u> : A Banach space E is said to possess the approximation property if for every compact subset K and every $\varepsilon > o$ there exists an operator $T \in J(E)$, such that

$$||x - Tx|| \leq \varepsilon \quad \text{whenever } x \in K.$$

1.4.3 <u>DEFINITION</u> : A Banach space E is said to possess the metric approximation property if for every compact subset K and every $\varepsilon > o$ there exists an operator $T \in J(E)$ such that $||T|| \leq 1$ and

$$||x - Tx|| \leq \varepsilon \quad \text{whenever } x \in K.$$

1.4.4 **DEFINITION** : Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of Banach algebras with field $\mathbb{K}$. Denote by ΣA_λ the class of all functions f defined on Λ with $f(\lambda) \in A_\lambda$ such that $|f|$ is defined by

$$|f| = \sup_{\lambda \in \Lambda} \| f(\lambda) \| < \infty$$

the above norm being in A_λ. Algebra operations are defined in ΣA_λ by the usual relations $(f+g)(\lambda) = f(\lambda) + g(\lambda)$, $(\alpha f)(\lambda) = \alpha(f(\lambda))$ $(\alpha \in \mathbb{K})$ and $(fg)(\lambda) = f(\lambda)\, g(\lambda)$. With these operations and $|f|$ as norm, ΣA_λ is easily seen to be a Banach algebra. This algebra is called the normed full direct sum of the algebras A_λ.

The following result about the spectral decomposition of a Hermitian operator is proved in Naimark [15], p.248-249.

1.4.5 **THEOREM** : For every Hermitian operator $A \in L(H)$ there exists a unique function $P : \mathbb{R} \longrightarrow L(H)$, called the spectral function of A, with the following properties :

(i) $P(\lambda)$ is a projection operator for every $\lambda \in \mathbb{R}$.

(ii) $P(\lambda)\, P(\mu) = P(\lambda)$ for $\lambda \leq \mu$.

(iii) $P(\lambda)\, T = TP(\lambda)$ for every $T \in L(H)$ such that $AT = TA$.

(iv) The function $\lambda \longrightarrow P(\lambda)x$ is continuous from the left for every $x \in H$.

(v) If $[a,b]$ is a closed interval containing in its interior the entire spectrum of the operator A and f is an arbitrary continuous complex-valued function on $[a,b]$, then $P(\lambda) = 0$ for $\lambda < a$, $P(\lambda) = 1$ for $\lambda > b$ and we have

$$(*) \quad f(A) = \int_a^b f(\lambda)\, d\, P(\lambda), \text{ in particular}$$

$$(**) \quad A = \int_a^b \lambda\, d\, P(\lambda)$$

where the integrals are of Riemann-Stieltjes type and exist with respect to the norm of $L(H)$. The formula $(**)$ is called the spectral decomposition of the operator A.

1.4.6 <u>DEFINITION</u> : Let E and F be real or complex normed spaces. An operator $T \in L(E,F)$ is said to be a compact (resp. precompact) operator, if it maps the closed unit sphere of E onto a relatively compact (resp. precompact or, synonymously, totally bounded) set.

<u>REMARK</u> : Since every relatively compact set is precompact, a compact operator is always a precompact operator. However, in case F is a Banach space these concepts are equivalent.

CHAPTER - II

COMPACT AND WEAKLY COMPLETELY CONTINUOUS ELEMENTS IN NORMED ALGEBRAS

2.1 INTRODUCTION : The notion of completely continuous elements of a commutative normed algebra was introduced by M. Freundlich [12], who defined them as those elements of the algebra for which the corresponding regular representations are compact operators. For $L(E)$, the Banach algebra of bounded linear operators on a Banach space E, K. Vala [24] proved that $T \in L(E)$ is a compact operator on E if and only if the map $X \longrightarrow TXT$ is a compact operator on $L(E)$. Motivated by this result he has defined in [25] that an element u of an arbitrary Banach algebra A is to be regarded as compact, if the map $x \longrightarrow uxu$ is a compact operator on A.

Subsequent investigations (see [1], [27], [28]) have further indicated that this definition indeed yields a natural extension of the notion of a compact operator. For a Hilbert space H, T. Ogasawara proved in [16] that $T \in L(H)$ is a compact operator if and only if the map $X \longrightarrow TX$ is a weakly compact operator on $L(H)$. In analogy to this, K. Ylinen [30] suggested another generalization of the concept of a compact operator in the context of C^*-algebras. For such an algebra A, he defined an element $u \in A$, to be weakly completely continuous, if the map

$x \longrightarrow ux$ (resp. $x \longrightarrow xu$) is a weakly compact operator on A.

The first section of this chapter is devoted to the study of completely continuous elements of a normed algebra in the sense of Freundlich [12]. In the second section compact elements of normed algebras in the sense of Vala [25] have been discussed. In addition we also deal with approximation numbers of such elements in the context of C^*-algebras. In the third section we present results about equivalence of compactness and weak complete continuity of elements showing that the two generalizations are same in case of C^*-algebras.

2.2 PRELIMINARIES : In this section a commutative Banach algebra over the field of complex numbers with identity element e will be denoted by A. An element a of A is said to be completely continuous (denoted as c.c.) if the operator T_a on A obtained by $T_a(x) = ax$ is c.c. This yield the following :

2.2.1 DEFINITION : $a \in A$ is c.c. if aB has compact closure for every bounded set $B \subset A$.

This is evidently equivalent to the condition that aS be relatively compact where S is the unit sphere. Even if S is closed, aS may not, in general be compact. For example, consider the Banach algebra ℓ_∞ of all bounded sequences of numbers under pointwise addition and multiplication and with sup norm. Let S be the closed unit sphere and take $a = (1,1, \ldots, 1, \ldots) \in \ell_\infty$. Then $aS = S$, which is not compact as ℓ_∞ is infinite-dimensional.

c.c. elements can not be units in an infinite-dimensional Banach algebra. In fact, we have the following :

2.2.2 **THEOREM** : [12] A unit is c.c. if and only if A is finite-dimensional.

PROOF : If A is finite-dimensional then S, the unit sphere in A, is compact since a Banach space is finite-dimensional if and only if the unit sphere in it is compact and hence every bounded set is relatively compact. Since multiplication preserves boundedness of subsets, every element is consequently c.c., and in particular, every unit is.

Conversely, if a unit r is c.c. and let S be the closed unit sphere. Then rS is relatively compact i.e. $\overline{rS}$ is compact. But $rS = \overline{rS}$. Hence rS is compact. Since multiplication is continuous, it follows that $r^{-1}(rS)$ is compact. Therefore, S is compact. Hence A is finite-dimensional.///

As in the case of c.c. operators, the c.c. elements also form a closed ideal in the algebra A. This follows from the fact that the difference of c.c. elements and the product of a c.c. element by an arbitrary element of A are c.c. Therefore, the set C of all c.c. elements of A is an ideal. By a theorem of Banach ([4], Th. 2, p. 96) the limit of a sequence of c.c. operators is a c.c. operator, and since A is a closed subalgebra of the algebra L(A) of bounded linear operators in A, the limit of a sequence of c.c. elements is again a c.c. element.

Given an ideal I let us denote by A_I the annihilator ideal of I, that is, the set of all elements of A which annihilate every element of I. If I is the principal ideal generated by the element x it is denoted by $\{x\}$ and its annihilator by A_x.

By the range of an element of A considered as a linear operator we simply mean the principal ideal $\{r\}$ generated by it. In this case $\{r\} \cap S$ is clearly the unit sphere of $\{r\}$ considered as a normed algebra (since $\{r\}$ may not be closed, this may lack completeness). If this is relatively compact in A then since $rS \subset \{r\} \cap S$, r is c.c.; if the sphere of $\{r\}$ is compact, $\{r\}$ is finite-dimensional, that is to say, r considered as an operator has a finite-dimensional range. This suggests the following :

2.2.3 <u>DEFINITION</u> : [12] r is finite-dimensional if $\{r\}$ is finite-dimensional.

Finite-dimensional elements are c.c. since the unit sphere in their range is compact but the converse is not true in general. For example, the element $x = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$ which is not a finite-dimensional element of the space ℓ_∞ of bounded sequences of numbers (multiplication is defined coordinatewise) may be thought of as the limit of the sequence $\{x_n\}$ where $x_n = (1, \frac{1}{2}, \ldots, \frac{1}{n}, o, o, \ldots)$ are all finite-dimensional. The x_n are all c.c. and so is their limit x.

A theorem of Banach ([4], Theorem 12, p. 152) translated in the context of above terminology gives the following result.

2.2.4 <u>THEOREM</u> : [12] If $a \in A$ is c.c. then A_{e-a} is finite dimensional.

From this it follows that if x^{n-1} is c.c. and $x = x^n$ then x is finite-dimensional. In particular, the following result is obtained.

2.2.5 <u>COROLLARY</u> : An idempotent which is c.c. is also finite-dimensional.

We also have the following :

2.2.6 <u>THEOREM</u> : If u is c.c. then $t = e - u$ is not nilpotent.

<u>PROOF</u> : Suppose that the theorem is false, then an integer n exists such that $t^n = o$. But then $t^n = (e-u)^n$ implies

that $o = t^n = e - u^*$, where u^* is c.c. as seen from the binomial expansion of $(e-u)^n$, whence $e = u^*$ and so e is c.c., which by Theorem 2.2.2, is impossible in an infinite dimensional Banach algebra.///

2.3.1 Let E_j $(j = 1,2,3,4)$ be given normed spaces; denote by $L(E_i, E_k)$ the normed space of all bounded linear mappings from E_i into E_k $(i, k = 1,2,3,4)$. Given three bounded linear mappings $A \in L(E_3, E_4)$, $T \in L(E_2, E_3)$ and $C \in L(E_1, E_2)$ one can form the composed mapping ATC :

$$E_1 \xrightarrow{C} E_2 \xrightarrow{T} E_3 \xrightarrow{A} E_4$$

Keeping A and C fixed and varying T in $L(E_2, E_3)$, we can write $ATC = u(T)$, where u is then regarded as a bounded linear mapping of $L(E_2, E_3)$ into $L(E_1, E_4)$. The mapping u depends on the mappings A and C and so their qualities are reflected in those of u. Particularly, K.Vala [24] has proved the following :

2.3.2 <u>THEOREM</u> : The mapping $T \longrightarrow u(T) = ATC$, where $A, C \neq 0$, is a precompact operator (in the uniform operator topology) if and only if both the mappings A and C are precompact.

By taking $E_i = E$ $(i = 1,2,3,4)$ and $C = A$ in the preceeding theorem we obtain also the following :

2.3.3 <u>THEOREM</u> : [25] The necessary and sufficient condition in order that the linear map $T \longrightarrow ATA$ from $L(E)$ into $L(E)$ be precompact is that the map A from E into itself be precompact.

These observations motivated K. Vala [25] to define compact and finite-dimensional elements in an arbitrary normed algebra as follows :

2.3.4 <u>DEFINITIONS</u> : An element a of a normed algebra A is said to be compact if the map $T_a : A \longrightarrow A$, defined by $T_a(x) = axa$ $(x \in A)$ is a precompact operator on A; a is said to be finite-dimensional if T_a is of finite rank. If T_a is of rank one, a is defined to be 1-dimensional.

<u>REMARK-1</u> : The above definition of a compact element is indeed a generalization of the notion of a compact operator, for if we take for A the full operator algebra $L(E)$ on a normed space E, Theorem 2.3.2 shows that compact elements of A are precisely the compact operators on E.

<u>REMARK-2</u> : It is evident from the definition that a compact (resp. finite-dimensional) element of A is a compact (resp. finite-dimensional) element of every subalgebra of A. This fact provides a justification for the definition of compact elements in terms of precompact instead of compact operators. In the case of Banach algebras this distinction is immaterial since precompact operators are the same as compact operators.

Let us denote by C the set of all compact elements and by F the set of all finite-dimensional elements of a normed algebra A respectively. Clearly we have $F \subset C$ and the following result holds.

2.3.5 **THEOREM** : [25] The set C is closed.

PROOF : Let $a \in \bar{C}$ and $\{a_k\} \subset C$ be a sequence such that $a_k \longrightarrow a$, as $k \longrightarrow \infty$. Now for any arbitrary $x \in A$, we have,

$$||a_k x a_k - axa|| = ||(a_k - a) x a_k + ax(a_k - a)||$$

$$\leq ||(a_k - a)x a_k|| + ||ax(a_k - a)||$$

$$\leq (||a_k - a|| \; ||a_k|| + ||a|| \; ||a_k - a||) \; ||x||$$

$\leq M \; ||a_k - a|| \; ||x||$, where M is a constant. From this it follows that $||T_{a_k} - T_a|| \longrightarrow 0$ as $k \longrightarrow \infty$. As T_{a_k}'s are precompact, it follows that T_a is precompact and hence by definition a is a compact element.///

This result implies that $\bar{F} \subset C$.

2.3.6 **THEOREM** : [25] (cf. Corollary 2.2.5) Every compact idempotent element of A is finite-dimensional.

PROOF : In fact, from $a^2 = a \Rightarrow a^2xa^2 = axa$ for each $x \in A$ or $T_{a^2} = T_a$; As the operator T_a is precompact $\Rightarrow$ it is of finite-rank.///

The following theorem proved by K. Vala [25] expresses the fact that the sets C and F are stable w.r. to multiplication in A.

2.3.7 <u>THEOREM</u> : If a is a compact (resp. finite-dimensional) element and u be an arbitrary element of A, then the elements au and ua are compact (resp. finite-dimensional) elements of A.

<u>PROOF</u> : The maps $x \longrightarrow aux\,au$ and $x \longrightarrow uaxua$ are both compositions of three operators one of which is T_a. Therefore, the theorem follows from the fact that the precompact operators (resp. of finite-rank) form a 2-sided ideal.///

2.3.8 <u>DEFINITION</u> : If $\{A_\lambda : \lambda \in \Lambda\}$ is a family of Banach algebras, then $(\Sigma A_\lambda)_o$ denotes the subset of the full direct sum ΣA_λ consisting of all elements f of ΣA_λ such that, for arbitrary $\varepsilon > o$, the set $\{\lambda : \|f(\lambda)\| \geq \varepsilon\}$ is finite. Then $(\Sigma A_\lambda)_o$ is a Banach algebra with respect to the norm $\|f\| = \sup\limits_{\lambda \in \Lambda} \|f(\lambda)\|$.

2.3.9 <u>DEFINITION</u> : [1] A Banach algebra A is said to be compact if each element $a \in A$ is compact.

It follows from Corollary 3.3 and Lemma 3.6 of [1] that the Banach algebras $(\sum\limits_{\lambda \in \Lambda} A_\lambda)_o$ are compact as long as A_λ's are chosen to be the Banach algebras of compact operators on a Hilbert space. In the following theorem it has been shown that compact Banach algebras arise precisely in this way.

2.3.10 <u>THEOREM</u> : [1] A compact B^*-algebra A is of the form $(\sum_{\lambda\varepsilon\Lambda} A_\lambda)_o$, where each A_λ is the algebra of all compact operators on a Hilbert space H_λ.

<u>PROOF</u> : Let $\{P_\lambda : \lambda \varepsilon \Lambda\}$ be the family of primitive ideals of A. Since the structure space of a compact Banach algebra is discrete by Theorem 6.1 in [1] therefore, the structure space of A is discrete as A is a compact B^*-algebra. By a theorem in ([21], Theorem 4.9.24, p. 259) which states that 'A B^*-algebra A has a discrete structure space if and only if it is of the form $(\Sigma A_\lambda)_o$, where each A_λ is a topologically simple B^*-algebra' we get $A = (\Sigma A_\lambda)_o$, where $A_\lambda = A/A_\lambda$. Since each A_λ is a primitive compact B^*-algebra, the result follows from ([1], Theorem 8.1) which says that for a Banach*-algebra A such that $||a||^2 \leq k\ ||a^*a\ ||$ for some constant k, if A is primitive and compact then there exists a *-homeomorphism of A onto the algebra of compact operators on some Hilbert space H. If A is a B^*-algebra, this homeomorphism is an isometry.///

The following is a consequence of the preceding theorem.

2.3.11 <u>COROLLARY</u> : A compact B^*-algebra is a dual B^*-algebra, and conversely.

<u>PROOF</u> : A compact B^*-algebra A is of the form $A = (\sum_{\lambda\varepsilon\Lambda} A_\lambda)_o$ in the statement of the preceding theorem and such an

algebra is a dual B^*-algebra by Theorem 4.10.25 and Corollary 4.10.20 in [21] which states that given a family $\{A_\lambda : \lambda \in \Lambda\}$ of dual B^*-algebras, the B^*-algebra $A = (\Sigma A_\lambda)_o$ is also dual and every topologically simple annihilator B^*-algebra is dual which is equal to the algebra of all compact operators on a Hilbert space, respectively.///

In the case of C^*-algebras, the following result has been obtained by K. Astala and M.S. Ramanujan ([3], Lemma 2.2).

2.3.12 <u>THEOREM</u> : Let A be a C^*-algebra. Then A is compact if and only if A is a dual C^*-algebra.

<u>PROOF</u> : Since the socle of a C^*-algebra A coincides with the set of finite-dimensional elements F of A ([27], Theorem 5.1) and $\bar{F} = C$, the set of compact elements of A ([27], Theorem 3.10), we get that the socle of A is dense in A if and only if A is compact. Because a C^*-algebra A is dual if and only if the socle of A is dense in A ([13], Theorem 2.1), the result follows.///

2.3.13 <u>DEFINITION</u> : Let A be a Banach algebra A non-zero element $u \in A$ is called one-dimensional, if there exists a linear functional f_u on A such that

$$uxu = \langle f_u, x \rangle u, \text{ for all } x \in A.$$

APPROXIMATION NUMBERS :

2.3.14 DEFINITION : For a given operator $S \in L(E)$, E a Banach space, the n-th approximation number $a_n(S)$ is defined by

$$a_n(S) = \inf\{||S-A|| : A \in L(E),\ \text{rank}\ (A) < n\}.$$

In analogy to this definition, for a given Banach algebra A the n-th approximation number of an element $u \in A$ is given by

$$a_n(u) = \inf\{||u - \sum_{i=1}^{k} u_i|| : u_i \in F_1,\ k < n\},$$

where F_1 is the class of 1-dimensional elements of A.

The following properties can be easily checked.

(i) $||u|| = a_1(u) \geq a_2(u) \geq \dots \geq 0$ for $u \in A$.

(ii) $a_n(u+v) \leq a_n(u) + ||v||$ for $u,v \in A$.

(iii) $a_n(tuv) \leq ||t||\ a_n(u)\ ||v||$ for $t,u,v \in A$.

2.3.15 Let A be a C^*-algebra and C the class of all compact elements of A. Let $u \in C$ and (λ_n) the eigenvalues of u^*u, arranged in decreasing order and repeated according to multiplicity. Put $s_n(u) = (\lambda_n)^{1/2}$. This number is called the n-th singular value of the element u. By Corollary 2.3.11, the ideal C is a dual C^*-algebra.

Moreover, if $u \varepsilon C$ is one-dimensional with respect to C, then because of $uxu = uxux_0u = f(xux_0)u$ for all $x \varepsilon A$, where f is a linear functional on A such that $uxu = \langle f, x \rangle u$ for all $x \varepsilon A$ and $x_0 \varepsilon C$ is chosen such that $f(x_0) = 1$, it follows that $u \varepsilon F_1$. Therefore all results proved by Wong [26] for singular values are valid for C. Particularly we have

$$a_n(u) = S_n(u), \text{ for } u \varepsilon C.$$

The following result has been proved by J. Puhl [20].

2.3.15 THEOREM : Every $s \varepsilon C$ admits a Schmidt representation, i.e.

$$s = \sum_{i=1}^{\infty} \lambda_i u_i ,$$

where $u_i \varepsilon F_1$, $||u_i|| = 1$, and $u_i u_i^*$ (resp. $u_i^* u_i$) are mutually orthogonal idempotents. Moreover, $a_n(s) = \lambda_n$.

PROOF : Since the C^*-algebra A is *-isomorphic to a uniformly closed *-subalgebra of $L(H)$, where H is a Hilbert space, therefore by polar decomposition of $s \varepsilon A$ we get $s = U|s|$ and $|s| = U^*s$, where $U \varepsilon L(H)$ is partially isometric. Since $s \varepsilon C$ implies that $|s| \varepsilon C$, and hence there exists the following spectral decomposition using a result of Ylinen ([27], Theorem 3.8) $|s| = \sum_{i=1}^{\infty} \lambda_i e_i$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$ and e_i's are orthogonal minimal hermitian idempotents.

We get,

$$s = \sum_{i=1}^{\infty} \lambda_i u_i,$$

where $u_i = Ue_i = (1/\lambda_i)se_i \in F_1$ and $||u_i|| \leq 1$. From $|s| = U^*U|s|$ it follows that $e_i = U^*Ue_i = U^*u_i$ and $1 \leq ||u_i||$. Therefore,

$$(u_i u_i^*)(u_i u_i^*) = u_i e_i U^* u_i u_i^* = u_i e_i u_i^* = u_i u_i^*$$

and

$$(u_i u_i^*)(u_j u_j^*) = u_i e_i U^* U e_j u_j^* = u_i e_i e_j u_j^* = o \text{ for } i \neq j.$$

Similarly it can be shown that $u_i^* u_i$ are orthogonal minimal idempotents. By definition $s_n(s) = \lambda_n$. Because of $a_n(s) = s_n(s)$ we get $a_n(s) = \lambda_n$.///

The following definitions will be used in the sequel.

2.4.1 <u>DEFINITION</u> : ([10], p. 67) Let X be a linear topological space. A set $A \subset X$ is said to be weakly sequentially compact if every sequence $\{x_n\}$ in A contains a subsequence which converges weakly to a point in X.

2.4.2 <u>DEFINITION</u> : ([10], p. 482) Let E, F be Banach spaces, $T \in L(E,F)$, and S be the closed unit sphere in E. The operator T is said to be weakly compact if the weak closure of TS is compact in the weak topology of F.

Thus an operator is weakly compact if and only if it maps bounded sets into weakly sequentially compact sets.

The following definition of weakly semi-completely continuous elements of Banach algebras was given by B.J. Tomiuk and P.K. Wong in [23].

2.4.3 DEFINITION : An element u of a Banach algebra A is called weakly semi-completely continuous (denoted as w.s.c.c.) if the map $x \longrightarrow uxu$, $x \in A$, is a weakly compact operator on A.

In order to prove the next theorem we will make use of the following result ([29], Lemma 3.1).

2.4.4 LEMMA : If u is a self-adjoint w.s.c.c. element of the C^*-algebra A, then $Sp_A(u)$ consists of a countable number of points and zero can be the only point of accumulation, where $Sp_A(u)$ stands for the spectrum of u w.r. to A_1, the C^*-algebra obtained by adjoining an identity to A.

2.4.5 THEOREM : ([29], Theorem 3.1) Let A be a C^*-algebra and $u \in A$. The operator $x \longrightarrow uxu$ on A is compact if and only if it is weakly compact.

PROOF : The weakly compact operators on A form a two-sided ideal of L(A) ([10], p. 484). It follows that y^*, xy, and yx are w.s.c.c. if $y \in A$ is w.s.c.c. and $x \in A$. Let $u \in A$ be w.s.c.c., then u^*u is a self-adjoint w.s.c.c. element of A. If A is realized as a sub-C^*-algebra of L(H) for a Hilbert space H, then $Sp'_A(u^*u) = Sp'_{L(H)}(u^*u)$ by ([9], p. 8), so that the spectrum of u^*u as an operator on H (Lemma 2.4.4) is countable and zero can be the only point of accumutation, the non-zero part in that spectrum being the same as in $Sp'_{L(H)}(u^*u)$. Therefore a representation $u^*u = \Sigma \lambda_n e_n$ is obtained using the method of the proof of Theorem 3.8 in [27], where each e_n is a w.s.c.c. projection, $\lambda_n > o$ and the series converges in norm. As the closed unit sphere of $e_n A e_n$ is $\sigma(A,A^*)$-compact and so $\sigma(e_n A e_n, (e_n A e_n)^*)$-compact, the C^*-algebra $e_n A e_n$ is reflexive ([10], p. 425) and hence finite-dimensional using a result of Ylinen ([29], Lemma 3.2). Hence Theorem 3.10 in [27] yields $|u| = (u^*u)^{1/2} = \Sigma \lambda_n^{1/2} e_n$ is a compact element of A, and similarly $|u^*| = (uu^*)^{1/2}$ is compact. Since the map $x \longrightarrow |u|x|u^*|$ is a compact operator on A using Theorem 3.9 in Ylinen [27] which states that, 'Let A be a *-subalgebra of L(H) and let S and T be compact (resp. finite-dimensional) elements of A. Then the operator $X \longrightarrow SXT$ on A is precompact (resp. has a finite-dimensional range).', so is the map $x \longrightarrow uxu = s|u|x|u^*|t^*$, where $u = s|u|$ and $u^* = t|u^*|$ are the polar decompositions of u and u^*.

The 'only if' part being easy, is left.///

The following definition of left (resp. right) weakly completely continuous elements in a Banach algebra was given by Ylinen [30].

2.4.6 <u>DEFINITION</u> : Let A be a Banach algebra, $u \in A$ is called a left (resp. right) weakly completely continuous (denoted as l.w.c.c. and r.w.c.c. respectively) element of A, if the map $x \longrightarrow ux$ (resp. $x \longrightarrow x\,u$) is a weakly compact operator on A.

2.4.7 <u>REMARK</u> : l.w.c.c. (resp. r.w.c.c.) elements of A form a closed two-sided ideal using Corollary 6 in ([10], p. 484) and in the context of C^*-algebras these ideals are thus selfadjoint ([9], p. 17), and so if an element u in a C^*-algebra A is l.w.c.c. it is also r.w.c.c. (the operator $x \longrightarrow (u^*x^*)^* = xu$ is weakly compact), and conversely. Therefore l.w.c.c. (resp. r.w.c.c.) elements of a C^*-algebra will be simply called weakly completely continuous (w.c.c.).

Now it can be shown that in the case of C^*-algebras an element is compact if and only if it is weakly completely continuous i.e. the two generalizations of a compact operator are in fact the same for C^*-algebras.

We need the following ([30], Corollary 2) in the proof of the next theorem.

2.4.8 <u>COROLLARY</u> : Let H be a Hilbert space and $T \in L(H)$. Then T is a compact operator on H if and only if T is a w.c.c. element of $L(H)$.

Now the main theorem of this section which has been proved by Ylinen [30] can be stated as follows :

2.4.9 <u>THEOREM:</u> Let A be a C^*-algebra and $u \in A$. The following three conditions are equivalent :

(i) the map $x \longrightarrow uxu$ is a compact operator on A.

(ii) the map $x \longrightarrow ux$ is a weakly compact operator on A.

(iii) the map $x \longrightarrow xu$ is a weakly compact operator on A.

<u>PROOF</u> : It is already noted in the Remark 2.4.7 that (ii) and (iii) are equivalent. Assume now (i). There is an isometric $*$-representation π of A on a Hilbert space H such that $\pi(u)$ is a compact operator on H by a Theorem in [28]. By Corollary 2.4.8, the operators $X \longrightarrow \pi(u)X$ and $X \longrightarrow X\pi(u)$ on $L(H)$ are weakly compact. Since $\pi(A)$ is $\sigma(L(H), L(H)^*)$-closed and the relative $\sigma(L(H), L(H)^*)$-topology on $\pi(A)$ agrees with $\sigma(\pi(A), \pi(A)^*)$, it follows that $x \longrightarrow ux$ and $x \longrightarrow xu$ are weakly compact operators on A.

Assume now (ii). As the ideal W of the w.c.c. elements is self adjoint, it is a sub-C^*-algebra of A. Since each element of W is w.c.c., W is a dual C^*-algebra by ([17], Theorem 6, p. 21). As W has an approximate identity ([9], p. 15), Cohen's factorization theorem ([7], Theorem 1) implies that $u = vw$ for some $v, w \in W$. Thus the operator $x \longrightarrow uxu$ on A may be written as $T_3T_2T_1$ where $T_1x = xv$, $x \in A$, $T_2y = wyw$, $y \in W$, and $T_3z = vz$, $z \in W$. But $T_2 : W \longrightarrow W$ is a compact operator (see e.g. Corollary 2.3.11), and thus (i) holds.///

CHAPTER III

SINGLE, FINITE AND NUCLEAR ELEMENTS IN BANACH ALGEBRAS

3.1 INTRODUCTION : An element s of a C^*-algebra A is called single if, whenever asb = o for some a,b in A, we have at least one of as, sb zero. For example it is easy to see that an operator of rank one is a single element of any algebra of operators that contains it.

Section 1 of this chapter is devoted to the study of single elements in the context of a C^*-algebra A, where it is shown that in the multiplicative semigroup of A the single elements form a selfadjoint semigroup ideal and if this ideal contains a non-zero element than it has a non-zero selfadjoint idempotent. Also an algebraic connection is shown between single elements and the operators of rank one. Stability of single elements under algebraic homomorphisms is also considered. Finally, if A contains a non-zero selfadjoint single element e, an inner product is defined on the elements of the ideal Ae and it is shown that this ideal equipped with this inner product is a Hilbert space.

In section 2 the notion of the trace of finite elements of a Banach algebra is introduced which includes the notion of the trace of operators. An element u of a complex, semi-prime Banach algebra M is said to be

1-dimensional if there exists a linear functional f_u on M such that, $uxu = \langle f_u, x \rangle u$ for all $x \in M$. For such elements there exists a unique complex number $tr(u)$ such that $u^2 = tr(u)\, u$, which is called the trace of u. A finite element u in the sense of K. Vala has a representation $u = \sum_1^m u_i$, where u_i is one-dimensional. Also by putting $tr(u) = \sum_1^n tr(u_i)$, a well-defined trace of u is obtained. Also the definition of nuclear elements is given in a natural way.

A Banach algebra A has the quasi-approximation property (denoted as q.a.p.) if for each minimal idempotent $q \in A$ the Banach space Aq (resp. qA) has the approximation property. It has been indicated that commutative Banach algebras and C^*-algebras have q.a.p. It is also shown that if the algebra fulfills q.a.p. then the trace admits an extension to the nuclear elements. Also the trace of a nilpotent nuclear element is shown to be zero. The trace formulas are also studied and the use of the well-known results regarding trace formulas of certain operator classes is made in order to obtain similar results in the case of Banach algebra.

3.2.1 <u>SINGLE ELEMENTS</u> : The set of single elements of a C^*-algebra A will be denoted by σ. It is easy to see that the zero element is single in A i.e. it is a member of σ.

We have the following result proved by J.A. Erdos [11].

3.2.2 **LEMMA** : If $s \in \sigma$ then $s^* \in \sigma$. For any element x in A, $xs \in \sigma$ and $sx \in \sigma$.

PROOF : Let $as^*b = o$, for some $a, b \in A$, then $(as^*b)^* = b^*sa^* = o$ and since s is single, we have, either $b^*s = o$ or $sa^* = o$. Hence either $(b^*s)^* = o$ or $(sa^*)^* = o$ i.e. either $s^*b = o$ or $as^* = o$ and so s^* is single by definition.

If $axsb = o$ and since s is single, we have either $axs = o$ or $sb = o$. Hence either axs or xsb is zero and thus xs is single. In a similar manner we can show that sx is single.///

From this lemma it is evident that in the multiplicative semigroup of A the single elements form a self-adjoint semigroup ideal. If this ideal contains a non-zero element s then it contains a non-zero self-adjoint element s^*s. In this case it is shown below that it will contain a non-zero self-adjoint idempotent.

3.2.3 **LEMMA** : If $s \in \sigma$ and s is normal then for some complex number λ, we have

(i) $s^2 = \lambda s$,

(ii) $s = \lambda e$, where e is a single self-adjoint idempotent.

PROOF : In case $s = o$ then $\lambda = o$ and $e = o$ may be chosen.

So let $s \neq o$ and let B be the commutative C^*-algebra generated by s and s^*. Then by the Gel'fand representation theorem B is isometrically $*$-isomorphic to the algebra $C_o(X)$ of all continuous functions vanishing at infinity on a locally compact Hausdorff space X. If $c \in B$ and let $\hat{c}$ be the image of c under the Gel'fand representation. We will show that the support of $\hat{s}$ (i.e. $\overline{\{x : \hat{s}(x) \neq o\}}$) has exactly one point. Since we have taken $s \neq o$ so $\hat{s} \neq o$ and hence the support of $\hat{s}$ must have at least one point. So let the support of $\hat{s}$ consists of two distinct points x_1 and x_2. Since a locally compact Hausdorff space is completely regular, thus there exists functions f and g in $C_o(X)$ with disjoint supports such that $f(x_1) \neq o$ and $g(x_2) \neq o$. Then we have $f\hat{s}g = o$ with $f\hat{s} \neq o$ and $\hat{s}g \neq o$. But since the Gel'fand representation is onto $C_o(X)$, there exist elements a and b in B such that $\hat{a} = f$ and $\hat{b} = g$. Then $asb = o$ but $as \neq o$ and $sb \neq o$. Which contradicts the fact that s is single. Hence $\hat{s}(x) = o$ except at one point x_o i.e. the support of s has exactly one point.

Let $\lambda = \hat{s}(x_o)$, then $(\hat{s})^2(x_o) = \lambda\, \hat{s}(x_o)$. Also if $\hat{e} = \lambda^{-1}\, \hat{s}$, we have, $\hat{e}\,(x_o) = \lambda^{-1}\, \hat{s}\,(x_o) = 1$ and $\hat{e}\,(x) = o$ for $x \neq x_o$. Thus $\hat{e}$ is a real idempotent function. Since the Gel'fand map is a $*$-isomorphism we get $s^2 = \lambda s$ and that $e = \lambda^{-1}s$ a self-adjoint single idempotent.///

3.2.4 <u>LEMMA</u> : For any single element s there exists self-adjoint single idempotents e and f such that $s = fse$.

PROOF : For $s = o$, $e = f = o$ can be chosen. Let us suppose that $s \neq o$. Using Lemma 3.2.3 we get single self-adjoint idempotents e and f such that for some non-zero complex numbers λ and μ, $s^*s = \lambda e$ and $ss^* = \mu f$. We show that $f = fse$. Since e is idempotent, $\lambda(se-s)s^*s = \lambda(se-s)\lambda e = \lambda^2 (se-s)e = \lambda^2 (se^2-se) = \lambda^2(se-se) = o$ and as s^* is single and $s^*s \neq o$, we have $(se-s)s^* = o$ by definition. Hence, $(se-s)(se-s)^* = (se-s)(se)^* - (se-s)s^* = (se-s)es^* - (se-s)s^* = o$, $\Rightarrow (se-s) = o$, using C^*-condition, i.e. $se = s$. Similarly it can be shown that $fs = s$. Therefore as e and f are idempotents,

$$s = se = fs = fse. ///$$

3.2.5 CORGLLARY : The principal left ideal As of a non-zero single element is equal to the principal left ideal of a single self-adjoint idempotent. Also $s \in As$. Similar statements hold for principal right ideal.

PROOF : If e is as in Lemma 3.2.4, then $As = Ase \subseteq Ae$. But $Ae = As^*s \subseteq As$. Hence, $As = Ae$ and as $s = se$, $s \in Ae = As$. ///

Now a connection between single elements and operaters of rank one is exhibited by the following.

3.2.6 THEOREM : [11] If s and t are single elements of A then the set

$$sAt = \{sat : a \in A\}$$

is a zero or one-dimensional linear subspace of A. Also $s \in sAs$.

<u>PROOF</u> : Obviously sAt is a linear subspace of A. If $sAt = (o)$ there is nothing to show and so we suppose that $sAt \neq (o)$. Let e and f be the single self-adjoint idempotents such that $ss^* = \mu f$ and $t^*t = \lambda e$. Then $s = fs$ and $t = te$. Hence,

$sAt = fsAte \subseteq fAe$ and $fAe = ss^*At^*t \subseteq sAt$.

Hence $sAt = fAe$ where e and f are single self-adjoint idempotents. If $s = t$ then $s = fse$ and $s \in sAs$.

Now let x and y be non-zero elements of sAt. Then $x = fxe$ and $y = fye$. Then $x \in Ae = Ay$ by Corollary 3.2.5. Hence for some $z \in A$, $x = zy$. It follows easily that

$$(*) \qquad x = fzfy.$$

If is shown that fAf consists of scalar multiples of f in order to complete the proof. If $a = a^*$ then faf is a self adjoint single element. By Lemma 3.2.3, $faf = \lambda g$, where g is a single self-adjoint idempotent. But clearly $g = fg = gf$ and so

$$(g-f)fg = gfg - fg = o$$

As f is single and $fg = g \neq o$, it follows that $(g-f)\,f = o$, which shows that $f = gf = g$.

Thus for any self-adjoint element a of A, $faf = \lambda f$. Since any element z of A may be written as $z = a + ib$, where a and b are self-adjoint, therefore

$$fzf = faf + ifbf = \lambda f + i\mu f.$$

Hence from $(*)$, if $k = \lambda + i\mu$, as $y = fy$,

$x = kfy = ky$ and the proof is complete.///

3.2.7 <u>COROLLARY</u> : If s is a non-zero single element of A, then As is a minimal left ideal and sA is a minimal right ideal.

<u>PROOF</u> : If L is a left ideal contained in As and ℓ is a non-zero element of L then $\ell = as$ for some $a \in A$ and

$$L \supseteq A\ell = Aas \supseteq As^*a^*as.$$

Using Corollary 3.2.5, $s^*s \in s^*As$ and as s^*As is one-dimensional,

$$s^*a^*as = ks^*s = \lambda e.$$

But $Ae = As$ and hence $L \subseteq As^*a^*as = As$. Thus As is a minimal left ideal. Similarly sA is a minimal right ideal.///

The stability of single elements under algebraic homomorphisms is indicated by the following :

3.2.8 <u>LEMMA</u> : If $s \in A$ is single and ϕ is any homomorphism of A into any algebra then $\phi(s)$ is single in $\phi(A)$.

PROOF : Let K be the kernel of Φ. Corollary 3.2.7 implies that As is a minimal left ideal. Hence either $As \cap K = As$ or $As \cap K = (o)$. If $\Phi(s) = o$ then it is single. If $\Phi(s) \neq o$ then since $s \in As$, $As \cap K = (o)$.

Now if $\Phi(asb) = o$ then $asb \in K$ and by the above, since sb is single, if $asb \neq o$ then $Asb \subseteq K$. But $sb \in Asb$ and so $\Phi(sb) = o$. If $asb = o$ then either as or sb is zero. So in either case we have $\Phi(as) = o$ or $\Phi(sb) = o$ and hence $\Phi(s)$ is single in $\Phi(A)$.///

Suppose that the C^*-algebra A contains a non-zero single element. Then by Lemma 3.2.3 it contains a non-zero self-adjoint single idempotent. The elements of the ideal Ae can be equipped with an inner product as follows :

If x and y belong to Ae then $x = xe$ and $y = ye$ and hence by Theorem 3.2.6 there exists a complex number $\lambda(x,y)$ such that

$$y^*x = ey^*xe = \lambda(x,y)e.$$

Now define

$$\langle x,y \rangle = \lambda(x,y).$$

which can be verified to satisfy the conditions of an inner product.

If $x \in Ae$, $\langle x, x\rangle e = x^*x = o$ and thus $\langle x,y\rangle$ is definite. In order to show positivity we note first that $\langle e,e\rangle = 1$

and that $\langle x,x \rangle = \overline{\langle x,x \rangle}$, thus $\langle x,x \rangle$ is real. If $\langle x,x \rangle$ takes negative values then there exists $y \in Ae$ such that $\langle y,y \rangle = -1$. By multiplying y by a complex number of modulus one we may arrange that the real part of $\langle y,e \rangle$ is zero. Then

$$\langle y+e,\ y+e \rangle = -1+1+2 \operatorname{Re} \langle y,e \rangle = 0,$$

which is a contradiction to the definiteness of $\langle x,y \rangle$.

In order to identify the inner product norm with the C^*-algebra norm, C^*-condition is to be used. Firstly we have,

$||e|| = ||e^2|| = ||ee^*|| = ||e||^2$ implying that $||e|| = 1$. Therefore, as $\langle x,x \rangle \geq 0$, $\langle x,x \rangle = \langle x,x \rangle ||e|| = ||\langle x,x \rangle e|| = ||x^*xe||$, and as $x = xe$,

$$||x^*xe|| = ||(xe)^* xe|| = ||ex^*xe|| = ||xe||^2 = ||x||^2.$$

Finally to show completeness it is now sufficient to prove that Ae is closed in A. Therefore, if $\{x_i\}$ is a sequence of elements of Ae converging to x,

$$x - xe = \lim_{i\to\infty} (x_i - x_i e) = 0.$$

Hence $x \in Ae$. Therefore, we have the following.

3.2.9 <u>THEOREM:</u> [11] The ideal Ae with the C^*-algebra norm and the inner product defined above is a Hilbert space.

Let us denote the set of finite sums of members of

σ by S and the closure of S by Σ. S can be identified with the socle of A. It is sufficient to prove the following result in view of Corollary 3.2.7.

3.2.10 LEMMA: [11] If M is a minimal left ideal of A then M = As for some single element s of A. A similar statement holds for right ideals.

PROOF : If $o \neq s \in M$, $As \subset M$. Since $s^*s \neq o$ and M is minimal, As = M. It remains to show that s is single. If asb = o and as ≠ o then as before Aas = M. Thus Mb = Aasb = (o) since asb = o and so sb = o. Hence s is single.

A similar proof holds for right ideals.///

3.2.11 THEOREM : ([1], [11]) The C^*-algebra A is dual if and only if Σ = A.

PROOF : Since the C^*-algebra is dual if and only if its socle is dense ([13], Theorem 2.5), the result follows immediately using Lemma 3.2.10.///

3.3.1 FINITE AND NUCLEAR ELEMENTS : In the sequel A will denote a complex, semi-prime Banach algebra throughout this section. As is well-known the following condition is equivalent for A to be semi-prime.

If uxu = o for all x ε A then u = o.

In the next definition the concept of a one-dimensional operator is generalized to algebras.

3.3.2 **DEFINITIONS** : [20] A non-zero element $u \in A$ is said to be one-dimensional, if there exists a linear functional f_u on A such that

$$uxu = \langle f_u, x \rangle u, \text{ for all } x \in A.$$

Also the trace of u, denoted by tr(u), is defined by

$$u^2 = tr(u)\ u.$$

For any $x_o \in A$ such that $ux_ou \neq o$ we get from

$$\langle f_u, x_ou \rangle u = ux_ouu = tr(u)\ ux_ou$$
$$= tr(u) \langle f_u, x_o \rangle u$$

implying that

$$tr(u) = \frac{\langle f_u, x_ou \rangle}{\langle f_u, x_o \rangle}.$$

If A has an identity then $tr(u) = \langle f_u, 1 \rangle$.

The set of all one-dimensional elements of A is denoted by F_1.

If A is commutative, then an element $u \in A$ is one-dimensional if and only if there exists a linear functional σ_u on A such that

$$ux = \langle \sigma_u, x \rangle u, \text{ for all } x \in A.$$

Since

$$\langle f_u, x \rangle u = uxu = u^2x = tr(u)\, ux$$
$$= tr(u) \langle \sigma_u , x \rangle u,$$

we get the identity,

$$\langle f_u, x \rangle = tr(u) \langle \sigma_u , x \rangle, \text{ for all } x \in A.$$

The following result shows that the concept of one-dimensional elements generalizes the notion of one-dimensional operators.

3.3.3 PROPOSITION : [20] Let $A = L(E)$ be the algebra of all bounded linear operators on a Banach space E. Then the one-dimensional elements of A are exactly the one-dimensional operators of $L(E)$. Moreover, $tr(a \otimes x) = \langle a, x \rangle$.

We also have the following :

3.3.4 DEFINITION : [20] The elements u of A of the form $u = \sum_{i=1}^{n} u_i$, where $u_i \in F_1$, are called finite.

The class of all finite elements of A is denoted by F.

3.3.5 LEMMA : [20] If $u \in A$ is a given non-zero element such that $\dim (uAu) < \infty$, then there exists a minimal idempotent $p \in Au$ (resp. $p \in uA$).

PROOF : Let $v \in uAu$ be a non-zero element such that $\dim(vAv)$ is as small as possible. Then for an arbitrary element $y \in A$ such that $vyv \neq o$, we have

$$(vyv)A(vyv) = vAv$$

Therefore, there exists an element $z \in A$ such that $vyv = vyvzvyv$. i.e. $(vzvyv-v)A(vzvyv-v) \subseteq vAv$. Because equality would imply

$$o = vy\,(vzvyv-v)A(vzvyv-v)\,yv = vyvAvyv$$
$$= vAv. \text{ Hence,}$$

$$(vzvyv-v)A(vzvyv-v) = o$$

since A is semi-prime, it follows that $v = vzvyv$.

Now it is shown that Av is a minimal left ideal. Let $I \subseteq Av$ be a non-zero left ideal. Then there exist $y_o \in I$, $x_o \in A$ with $y_o x_o y_o \neq o$ and $y_o = yv$. An element $z \in A$ can be found such that $v = vzvx_o yv$. Therefore, $Av \subseteq Avx_o yv \subseteq I$ i.e. Av is a minimal left ideal. Hence there exists a minimal idempotent $p \in Av$ Au.///

The following result is obtained using preceeding Lemma.

3.3.6 <u>COROLLARY</u> : A non-zero element u of A is one-dimensional i.e. $u \in F_1$ if and only if dim $(uAu) = 1$.

<u>PROOF</u> : If $u \in F_1$ then clearly dim $(uAu) = 1$.

For converse part, let dim $(uAu) = 1$. Then using Lemma 3.3.5, there exists a minimal idempotent $p \in Au$. Then we have

$$(u-up)A(u-up) \subseteq uAu$$

since equality gives $uAup = (o)$, which is false. Thus we get

$$(u-up)A(u-up) = (o),$$

since A is semi-prime, we get $u = up \in F_1$.///

J. Puhl [20] has proved the following :

3.3.7 THEOREM : Let A be a complex, semi-prime Banach algebra and let $u \in A$ be a non-zero element such that $\dim (uAu) < \infty$. Then there exists an idempotent $p \in F \cap uA$ and $pu = u$ (resp. $up = u$).

J.C. Alexander [1] has also proved the following.

3.3.8 COROLLARY : Let u be a non-zero element of a complex, semi-prime Banach algebra A. Then the operator $x \longrightarrow uxu$ has a finite rank if and only if $u \in F$.

PROOF: Let $u \in F$ then using Lemma 3.3.11 it immediately follows that the operator $x \longrightarrow uxu$ has a finite rank. For converse, let the operator $x \longrightarrow uxu$ has a finite rank, then by Theorem 3.3.7 there exists an idempotent $p \in F$ such that $u = pu \in F$.///

After introducing the notion of a trace of finite element in an algebra we indicate many results that can be assumed similar to well-known results of the classical operator theory.

3.3.9 DEFINITION : [20] Two elements $u,v \in F_1$ are called equivalent (denoted as $u \sim v$) if there exists some $x_o \in A$ such that $ux_ov \neq o$.

3.3.10 **LEMMA** : [20] The relation $\backsim$ is an equivalence relation on F_1.

PROOF : As reflexive and symmetric properties are easily verified, only transitivity will be shown. Let $u, v, w \in F_1$, $u \backsim v$ and $v \backsim w$. There exist x_o and x_1 such that $u\, x_o v \neq o$ and $v x_1 w \neq o$. By 3.3.1 there is $y_o \in A$ such that

$$o \neq (ux_o v) y_o (ux_o v) = \langle f_u, y_o ux_o \rangle \, ux_o v.$$

Hence, $o \neq \langle f_v, y_o ux_o \rangle \, vx_1 w = vy_o ux_o vx_1 w$. Thus, $ux_o vx_1 w \neq o$, so $u \backsim w$.///

3.3.11 **LEMMA** : [20] Let $u, v \in F_1$ and $u \backsim v$. Then the operator

$$D_{u,v}\, x = uxv = \frac{\langle f_u, xvy_o \rangle}{\langle f_v, y_o ux_o \rangle} \, ux_o v$$

is one-dimensional, $||D_{u,v}|| \leq ||u||\ ||v||$, and trace $D_{u,v} = tr\,(u)\ tr(v)$.

PROOF : Let $ux_o v = o$ for some $x_o \in A$.. Then there is $y_o \in A$ such that,

$$o \neq (ux_o v) y_o (yx_o v) = \langle f_v, y_o ux_o \rangle \, ux_o v.$$

Hence

$$D_{u,v}\, x = \frac{1}{\langle f_v, y\, ux_o \rangle} \, ux_o v y_o ux_o v$$

$$= \frac{\langle f_u, xvy_o \rangle}{\langle f_v, y_o ux_o \rangle} ux_o v.$$

Consequently, $D_{u,v}$ is one-dimensional. Also it follows from

$$\frac{\langle f_u, ux_o vvy_o \rangle}{\langle f_v, y_o ux_o \rangle} ux_o v = uux_o vv$$

$= tr(u)\ tr(v) ux_o v$, that

trace $D_{u,v} = tr(u)\ tr(v)$.

Since, $D_{u,v}\ x = uxv \Rightarrow ||D_{u,v}|| = \sup_{||x|| \leq 1} ||D_{u,v} x||$

$= \sup\ ||uxv|| \leq \sup_{||x|| \leq 1} ||u||\ ||x||\ ||v|| \leq ||u||\ ||v||$

$\Rightarrow ||D_{u,v}|| \leq ||u||\ ||v||$.///

3.3.12 **LEMMA** : [20] Let $u_i \in F_1$ such that $\sum_{i=1}^{m} u_i = o$.

Then $\sum_{i=1}^{m} tr(u_i) = o$.

PROOF : A disjoint decomposition A_k of $\{1, 2, \ldots, n\}$ is induced by the equivalence relation $\backsim$ on F_1. Thus for fixed k we get

$$o = \left(\sum_{i=1}^{m} u_i\right) x \left(\sum_{j \in A_k} u_j\right) = \sum_{i,j \in A_k} u_i x u_j \text{ for all } x \in A.$$

It follows using Lemma 3.3.11 that

$$o = \sum_{i,j \in A_k} tr(u_i)\ tr(u_j) = \left(\sum_{i \in A_k} tr(u_i)\right)^2$$

Hence, $\sum_{i=1}^{n} tr(u_i) = \sum_{k} \sum_{i \in A_k} tr(u_i) = o.$ ///

In view of the preceding Lemma a well-defined trace for finite elements can be introduced as follows :

3.3.13 <u>DEFINITION</u> : [20] Let $u = \sum_{i=1}^{n} u_i$, $u_i \in F_1$, be any representation of $u \in F$. Then

$$tr(u) = \sum_{i=1}^{n} tr(u_i)$$

is called the trace of u.

3.3.14 <u>THEOREM</u> : [20] Let A be a semi-prime Banach algebra. Then the trace has the following properties :

(i) The trace is a linear functional on F.

(ii) If $u \in F$ and $x \in A$, then $tr(ux) = tr(xu)$.

(iii) If $u \in F$ is nilpotent, then $tr(u) = o$.

(iv) Let A be an algebra with involution and $u \in F$. Then $tr(u^*) = \overline{tr(u)}$.

(v) Let A be a C^*-algebra. Then f_{vv^*} is a positive functional on A for all $v \in F_1$.

<u>PROOF</u> : (i) is obvious.

(ii) For $v \in F_1$ and $x \in A$ we have

$$\langle f_v, x \rangle xv = xvxv = tr(xv)xv$$

and $\langle f_v, x \rangle vx = vxvx = tr(vx)\, vx.$

Thus, $tr\,(xv) = tr(vx)$. Now it follows that $tr(xu) = \sum_{i=1}^{n} tr(xu_i) = \sum_{i=1}^{n} tr\,(u_i x) = tr(ux)$.

(iii) Let A_k be the disjoint decomposition of $\{1, 2, \ldots, n\}$ induced by the equivalence relation $\sim$ on F_1. Let us put

$$D_k x = ux\Big(\sum_{i \in A_k} u_i\Big) = \sum_{i,j \in A_k} u_i\, xu_j$$

Since $u \in F$ is nilpotent the finite-dimensional operator D_k is nilpotent, therefore trace $D_k = 0$. Using Lemma 3.3.11, on the other hand we get,

$$\text{trace } D_k = \sum_{i,j \in A_k} tr(u_i)\, tr(u_j) = \Big(\sum_{i \in A_k} tr(u_i)\Big)^2.$$

Hence $tr(u) = \sum_k \sum_{i \in A_k} tr(u_i) = 0.$

(iv) $v \in F_1$ implies that there exists a linear functional f_v on A such that $vxv = \langle f_v, x \rangle v$ for all $x \in A$. Since A is an algebra with involution, we have $(vxv)^* = (\langle f_v, x\rangle v)^*$ i.e. $v^* x^* v^* = \overline{\langle f_v, x \rangle}\, v^* = \langle f_v, x^* \rangle v^*$. Hence $v^* \in F_1$. Since $v^2 = tr\,(v)v$ implies that $(v^2)^* = (tr(v)v)^* = \overline{tr(v)}v^*$. Also $(v^*)^2 = tr(v^*)v^*$ and as $(v^2)^* = (v^*)^2$ we get, $tr(v^*) = \overline{tr(v)}$.

Hence, $tr(u^*) = \sum_{i=1}^{n} tr(u_i^*) = \sum_{i=1}^{n} \overline{tr(u_i)} = \overline{tr(u)}$ for $u \in F$.

(v) As we have $(vv^*x^*x)^2 = tr(vv^*x^*x)vv^*x^*x$

$\Rightarrow (vv^*x^*x)(vv^*x^*x) = tr(vv^*x^*x)vv^*x^*x$

$\Rightarrow (vv^*x^*xvv^*)x^*x = tr(vv^*x^*x)vv^*x^*x$

$\Rightarrow (\langle f_{vv^*}, x^*x \rangle vv^*)x^*x = tr(vv^*x^*x)vv^*x^*x$

$\Rightarrow \langle f_{vv^*}, x^*x \rangle vv^*x^*x = tr(vv^*x^*x)vv^*x^*x$

$\Rightarrow \langle f_{vv^*}, x^*x \rangle = tr(vv^*x^*x) = tr(xv(xv)^*)$

and since for any $u \in F_1$ we have by Lemma 2.8 in [20], $Sp_A(u) = \{o, tr(u)\}$, result follows because $tr(xv(xv)^*) \in Sp(xv(xv)^*)$.///

3.3.15 **DEFINITION** : Let $A = L(E)$ be the algebra of all bounded linear operators on the Banach space E. An operator $U \in L(E)$ is said to be nuclear if $U = \sum_{i=1}^{\infty} a_i \otimes y_i$, where $a_i \in E$, $y_i \in E$, and $\sum_{i=1}^{\infty} ||a_i||\ ||y_i|| < \infty$.

Generalizing the notion of nuclear operators, the concept of nuclear elements in the context of a semi-prime Banach algebra A was introduced by J. Puhl [20]. It is as follows :

3.3.16 **DEFINITION** : An element $u \in A$ is called nuclear if $u = \sum_{i=1}^{\infty} u_i$, where $u_i \in F_1$ and $\sum_{i=1}^{\infty} ||u_i|| < \infty$.

The class of all nuclear elements is denoted by N and we put $\nu(u) = \inf \sum_{i=1}^{\infty} ||u_i||$, where infimum is taken over all representations.

Certain interesting properties of nuclear elements are indicated by the following :

3.3.17 <u>THEOREM</u> : [20] N is a two-sided ideal of A with $F \subset N$ and ν is a norm on N such that $\nu(xuy) \leq ||x||\ \nu(u)\,||y||$ for all $x, y \in A$, $u \in N$. Moreover, N is complete with respect to this norm.

3.3.18 <u>EXAMPLE</u> : (i) Let A be the algebra $L(E)$. Then $T \in A$ is nuclear if and only if T is a nuclear operator.

(ii) Let A be the algebra ℓ_∞. Then $N = \ell_1$.

3.3.19 <u>THEOREM</u> : [20] If $u \in N$, then the operator $x \longrightarrow uxu$ is nuclear.

<u>PROOF</u> : Let us choose a representation

$u = \sum_{i=1}^{\infty} u_i$, $\sum_{i=1}^{\infty} ||u_i|| < \infty$, $u_i \in F_1$. Since

$D_u\, x = uxu = \sum_{i,j=1}^{\infty} u_i\, x\, u_j = \sum_{i,j=1}^{\infty} \langle a_{ij}, x \rangle\, u_{ij}$ and

$||a_{ij}||\ ||u_{ij}|| \leq ||u_i||\ ||u_j||$. It follows that

$\nu(D_u) \leq \sum_{i,j=1}^{\infty} ||u_i||\ ||u_j|| = \left(\sum_{i=1}^{\infty} ||u_i|| \right)^2 < \infty$. Hence

D_u is a nuclear operator.///

3.3.20 <u>COUNTER EXAMPLE</u> : Converse of the preceeding Theorem is not true even in the case of C^*-algebras. Indeed, let $A = \ell_\infty$. Then $u = (\frac{1}{n}) \in \ell_\infty$ and $u \in N$ by Example 3.2.18 (ii) but the operator $x \longrightarrow uxu = ((\frac{1}{n})^2 \xi_n)$, where $x = (\xi_n)$, is nuclear.

Now certain conditions under which the trace admits an extension to the nuclear elements are studied.

3.3.21 **THEOREM** : Let A be a semi-prime Banach algebra having the approximation property. Then every $u \in N$ has a well-defined trace.

PROOF : Let $o = \sum_{i=1}^{\infty} u_i$, $u_i \in F_1$ and $\sum_{i=1}^{\infty} \|u_i\| < \infty$. Let A_k be the disjoint decomposition of $\{1, 2, \ldots, n\}$ induced by the equivalence relation $\backsim$ on F_1 such that for $u, v \in F_1$, $u \backsim v$ if there exists some $x_o \in A$ such that $ux_ov \neq o$. Let us put

$$D_k\, x = \sum_{i,j \in A_k} u_i x u_j = (\sum_{i=1}^{\infty} u_i) x (\sum_{j \in A_k} u_j) = o.$$

Since $\nu(D_k) \leq (\sum_{i \in A_k} \|u_i\|)^2$ and by virtue of the approximation property it follows that

$$o = \operatorname{trace} D_k = (\sum_{i \in A_k} \operatorname{tr}(u_i))^2. \quad \text{Hence } \sum_{i=1}^{\infty} \operatorname{tr}(u_i) = o. ///$$

3.3.22 **THEOREM** : [20] Let A be a semi-prime Banach algebra having the following property :

Given $u \in F$ and $\varepsilon > o$, then there exists $x \in F$, $||x|| \leq 1+\varepsilon$ such that $xu = u$ or $ux = u$.

Then every $u \in N$ has a well-defined trace.

PROOF : For a given $v \in F$ and $\varepsilon > o$, there is $x \in F$, $x = \sum_{i=1}^{n} x_i$, $x_i \in F_1$, $||x|| \leq 1+\varepsilon$, such that $xv = v$. If $v = \sum_{i=1} v_i$ is a nuclear representation, with

$$\nu(v) + \varepsilon \geq \sum_{i=1}^{\infty} ||v_i||, \text{ then}$$

$$|tr(v)| = |tr(xv) = |\sum_{i=1}^{n} < f_{x_i}, v > | =$$

$$|\sum_{i=1}^{n} \sum_{j=1}^{\infty} < f_{x_i}, v_j >| \leq \sum_{j=1}^{\infty} |tr(x\, v_j)|$$

$$\leq (\sum_{j=1}^{\infty} ||v_j||) \; ||x|| \leq (\nu(v) + \varepsilon) \; (1+\varepsilon).$$

Hence $|tr(v)| \leq \nu(v)$ for $v \in F$. Therefore, the functional $v \longrightarrow tr(v)$ admits a unique extension on N. Also, if $u = \sum_{i} u_i$ is any nuclear representation, then

$$tr(u) = \lim_{n} tr(\sum_{i=1}^{n} u_i) = \sum_{i=1}^{\infty} tr(u_i). ///$$

3.3.23 **THEOREM** : [20] Suppose A has q.a.p.,

(i) Then every $u \in N$ has a well-defined trace.

(ii) If $u \in N$ is nilpotent, then $tr(u) = o$.

PROOF : Without loss of generality let us assume that A_q

has the approximation property for each minimal idempotent $q \in A$.

(i) For a given $u \in N$ let us choose a nuclear representation $u = \sum_{i=1}^{\infty} u_i$, with $\sum_{i=1}^{\infty} ||u_i|| < \infty$. Thus, the series $\sum_i tr(u_i)$ must be convergent. It is sufficient to show that from $\sum_{i=1}^{\infty} v_i = o$, where $v_i \in F_1$ and $\sum_{i=1}^{\infty} ||v_i|| < \infty$, it follows that $\sum_{i=1}^{\infty} tr(v_i) = o$.

The equivalence relation $\smile$ on F_1 defined by $v_i \smile v_j$ if there exists an $x_o \in A$ such that $v_i\, x_o\, v_j \neq o$, induces a disjoint decomposition A_k of the natural numbers such that $v_i \smile v_j$ for $i\ j \in A_k$. Let us choose minimal idempotents q_k such that $q_k \smile v_i$ for $i \in A_k$. Let $w_k := \sum_{i \in A_k} v_i$ and define an operator $L_k \in L(Aq_k)$ by $L_k x := w_k x$. It is to be noted that $w_k \in N$. An application of Lemma 3.2.11 shows that $L_k \in N(Aq_k)$ and trace $L_k = \sum_{i \in A_k} tr\,(v_i)$. Since $o = (\sum_{i=1}^{\infty} v_i)\, x\, w_k = w_k x w_k$, we get $w_k = o$. Therefore, $\sum_{i \in A_k} tr(v_i) = o$ and hence we get $o = \sum_k tr(w_k) = \sum_{i=1}^{\infty} tr(v_i)$

(ii) If $u \in N$ is nilpotent, then $L_k \in N(Aq_k)$ is so. Hence o = trace L_k and $tr(u) = o$.///

In order to establish the trace formula for finite elements in Banach algebras and the Lidskij trace formula for nuclear elements in C^*-algebras, we require many additional concepts which are given as under :

3.3.24 <u>DEFINITION</u> : (i) Let U be a left (right) ideal of A contained in F. The cardinality of any maximal orthogonal set of minimal idempotents in U (denoted by θ (U)) is called the order of U.

(ii) For any bounded operator T on A the null space (denoted by NUL (T)) is defined by

$$\text{NUL}\ (T) = \{y \in A : Ty = o\}.$$

(iii) The smallest non-negative integer n such that

$$\text{NUL}\ (T^{n+1}) = \text{NUL}\ (T^{n})$$

or $+\infty$ if no such n exists, is called the ascent of T (denoted by $\alpha(T)$).

(iv) The left (right) multiplication operator on A determined by $\lambda - u$ is the operator which takes $x \in A$ into $\lambda x - ux \in A$ (resp. $\lambda x - xu \in A$) and is denoted by $L_{\lambda-u}$ (resp. $R_{\lambda-u}$).

The following result can be found in ([5], p. 499).

3.3.25 <u>LEMMA</u> : Let $u \in N$ and $\lambda \in Sp\ (u)$. Then $\alpha := \alpha(L_{\lambda-u}) = \alpha\ (R_{\lambda-u}) < \infty$ and θ (NUL $(L^{\alpha}_{\lambda-u})) < \infty$.

3.3.26 <u>LEMMA</u> : [20] Let $A = L(E)$ be the algebra of all bounded linear operators on a Banach space E. Let $S \in A$ and $\lambda \in \mathbb{C}$ be an eigenvalue of S with finite algebraic multiplicity. The θ (NUL $(L^{\alpha}_{\lambda-S})$) is equal to the algebraic multiplicity of the eigenvalue λ of the operator S.

PROOF : Suppose n to be the algebraic multiplicity of the eigenvalue λ, i.e.

$$n = \dim \{x \in E : (\lambda I-S)^{\beta} x = o\}$$

for a certain natural number β.

Let $x_1, x_2, \ldots, x_n$ be any basis of the subspace $\{x \in E : (\lambda I-S)^{\beta} x = o\}$. There exist functionals $a_1, a_2, \ldots, a_n$ on E such that $\langle x_i, a_k \rangle = \delta_{ik}$. Let us put $P_i :=$ $a_i \otimes x_i$. Then $P_i \in \mathrm{NUL}\,(L^{\beta}_{\lambda-S})$ are mutually orthogonal minimal idempotents of $L(E)$. Hence,

$$n \leq \theta\,(\mathrm{NUL}\,(L^{\beta}_{\lambda-S})) \leq \theta\,(\mathrm{NUL}\,(L^{\alpha}_{\lambda-S})).$$

Secondly, if $Q_j \in \mathrm{NUL}\,(L^{\alpha}_{\lambda-S})$ are a maximal system of mutually orthogonal, idempotents, then $Q_j = b_j \otimes y_j$, where $b_j \in E'$, $y_j \in E$, and $(\lambda I-S)\, y_j = o$. Because the elements y_j are linearly independent we get, $\theta\,(\mathrm{NUL}\,(L^{\alpha}_{\lambda-S})) \leq n$.///

NOTATION : Let us put $n(\lambda,u) := \theta\,(\mathrm{NUL}\,(L^{\alpha}_{\lambda-u}))$. Then $n(\lambda,u)$ is said to be the algebraic multiplicity of the eigenvalue $\lambda \in Sp_A(u)$.

The proof of the following lemma can be given in the same way as for Theorem 3.4 in [20].

3.3.27 LEMMA : [20] Let $u \in N$ and $\lambda \in Sp(u)$, $\lambda \neq o$. Then there exists an idempotent $p(\lambda) \in \mathrm{NUL}\,(L^{\alpha}_{\lambda-u})$, $p(\lambda) = \sum_{i=1}^{n(\lambda,u)} p_i(\lambda)$, $p_i(\lambda)$ being mutually orthogonal idempotents such that $p(\lambda)x = x$ for all $x \in NUL\,(L^{\alpha}_{\lambda-u})$.

Let $n(\lambda, L_k)$ imply the algebraic multiplicity of the eigenvalue λ of the operator L_k. If λ does not belong to the spectrum of L_k, then $n(\lambda, L_k)$ is taken to be equal to zero.

3.3.28 <u>LEMMA</u> : [20] Let $u \in N$ and $\lambda \in Sp\,(u)$, $\lambda = o$. Then $n(\lambda, u) = \sum_k n(\lambda, L_k)$.

<u>PROOF</u> : Let $u \in N$ and $u = \sum_{i=1}^{\infty} u_i$ with $\sum_{i=1}^{\infty} \|u_i\| < \infty$ be any representation of u. By preceding lemma there is an idempotent $p(\lambda) \in NUL\,(L^{\alpha}_{\lambda-u})$ such that $p(\lambda) = \sum_{i=1}^{n(\lambda,u)} p_i(\lambda)$ and $p(\lambda)x = x$ for all $x \in NUL\,(L^{\alpha}_{\lambda-u})$. For fixed K, let us put $B_k := \{i : p_i \smile q_k\}$ (B_k may be empty). We get $p(\lambda) = \sum_k \sum_{i \in B_k} p_i(\lambda)$ and $o = (\lambda-u)^{\alpha} p_i(\lambda) = (\lambda-w_k)^{\alpha} p_i(\lambda)$ for $i \in B_k$. There exist $x_i \in A$ such that $p_i(\lambda)\, x_i\, q_k \neq o$. Since $p_i(\lambda)\, x_i q_k \in Aq_k$ are linearly independent, it follows that

$$n(\lambda, L_k) \geq \text{Card}\,(B_k)$$

Secondly, if for any natural number β there exists $z \in Aq_k$, $z = xq_k$, such that

$$o = (\lambda I - L_k)^{\beta} z = (\lambda - w_k)^{\beta} z = (\lambda - u)^{\beta} z,$$

then by applying Lemma 3.3.11 we have

$$z = p(\lambda)z = \sum_{i \in B_k} p_i(\lambda) x q_k = \sum_{i \in B_k} \frac{\langle f_{p_i(\lambda)}, xq_k y_o \rangle}{\langle f_{q_k}, y_o p_i(\lambda) x_i \rangle} p_i(\lambda) x_i q_k$$

Thus, we get $n(\lambda, L_k) \leq \mathrm{Card}\,(B_k)$ and therefore we get

$$n(\lambda,u) = \sum_k \mathrm{Card}\,(B_k) = \sum_k n(\lambda,L_k). ///$$

3.3.29 <u>THEOREM</u> : [20] (<u>TRACE FORMULA FOR FINITE ELEMENTS</u>) Let $u \in F$. Then $tr(u) = \sum_i \lambda_i n(\lambda_i, u)$.

<u>PROOF</u> : Since $L_k \in L(Aq_k)$ is a finite-dimensional operator, we get

$$\mathrm{trace}\, L_k = \sum_i \lambda_i\, n(\lambda_i, L_k).$$

On the other hand Lemma 3.3.11 gives

$$\mathrm{trace}\, L_k = tr(w_k).$$

Using preceding lemma, we get

$$tr(u) = \sum_k tr(w_k) = \sum_k \sum_i \lambda_i\, n(\lambda_i,L_k) = \sum_i \lambda_i\, n(\lambda_i,u). ///$$

3.3.30 <u>THEOREM</u> : [20] (<u>LIDSKIJ TRACE FORMULA</u>) Suppose A to be a C^*-algebra and let $u \in N$. Then $tr(u) = \sum_i \lambda_i\, n(\lambda_i, u)$.

<u>PROOF</u> : For a fixed k, it is easy to see using Theorem 3.3.14 (v) that

$$(x,y) = \langle f_{q_k q_k^*}, y^* x \rangle \text{ for } x,y \in Aq_k$$

defines an inner product on Aq_k. Also from

$$||xq_k q_k^*||^2 = ||q_k q_k^* x^* x\, q_k q_k^*|| = \langle f_{q_k q_k^*}, x^* x \rangle ||q_k q_k^*||$$

and

$$\frac{||xq_k q_k^*||}{||q_k||} \leq ||xq|| = \frac{||xq_k q_k^* q_k||}{|\langle f_{q_k}, q_k^*\rangle|} \leq \frac{||xq_k q_k^*||}{||q_k||}$$

it follows that

$(x,x) = ||x||^2$ for all $x \in Aq_k$. Hence Aq_k is a Hilbert space. Also $L_k \in N(Aq_k)$. Lidskij trace formula yields,

trace $L_k = \sum_i \lambda_i \; n(\lambda_i, L_k)$. Since trace $L_k = \sum_{i \in A_k} tr(u_i)$ exists, it follows from Lemma 3.3.11 that there also exists $tr(w_k)$ and trace $L_k = tr(w_k)$. Also, the representation is unique. This is an easy consequence of the fact that $uxw_k = w_k x w_k$ for $x \in A$. We get,

$tr(u) = \sum_k tr(w_k) = \sum_k \sum_i \lambda_i \; n(\lambda_i, L_k) = \sum_i \lambda_i n(\lambda_i, u)$ by using Lemma 3.3.28 and $\sum_k |tr(w_k)| \leq \sum_i ||u_i|| < \infty$.///

CHAPTER - IV

REPRESENTATIONS ON C^*-ALGEBRAS

4.1 INTRODUCTION : It is well-known that a C^*-algebra A can be faithfully represented as an algebra of operators on some Hilbert space H. Since the rank of the image of a given element of A may vary under different representations, therefore the problem of finding conditions for an element to have an operator of known rank as its image under some faithful representation of A assumes much significance.

Section 1 of this chapter is devoted to the result which was proved by J.A. Erdos [11] indicating that there exists a faithful representation of A such that the image of every non-zero single element of A is an operator of rank one. In Section 2, the above result is utilized to prove a theorem due to K. Ylinen [28] which guarantees the existence of a faithful representation π of A on some Hilbert space H such that an element u of A is compact in the sense of Vala [25] if and only if $\pi(u)$ is a compact operator on H. Section 3 of this chapter is devoted to a theorem due to A.H. Al-Moajil [2] which states that there exists an *-isometry of a primitive B^*-algebra A into the algebra of bounded linear operators on some Hilbert space H, such that C, the set of all compact elements of A, maps onto the algebra of compact operators on H.

In Section 4 of this chapter results exhibiting the existence of an isometric $*$-representation π of a C^*-algebra A on some Hilbert space H (called by Astala and Ramanujan [3] as an Erdos representation) have been discussed. It has been shown that an element b of A has an operator of rank one as its image under π if and only if b is a 1-dimensional element. Moreover, the concept of approximation numbers is also taken up and it has been indicated that approximation numbers corresponding to a compact element in A coincides with such numbers of the image under Erdos representation.

4.2.1 The following notations will be used in the sequel.

The set of finite sums of single elements of a C^*-algebra A, which forms an $*$-ideal (Lemma 3.2.2), will be denoted by S and its closure by Σ. For any non-zero single idempotent e we put He for the ideal Ae considered as a Hilbert space. Let us denote the set $AeA = \{aeb : a, b \in A\}$ by σ_e, the set of finite sums of elements of σ_e by Se and the closure of Se by Σ_e. The operator $x \longrightarrow \langle x, p \rangle q$ will be devoted by $p \otimes q$. Obviously S_e and Σ_e are two-sided $*$-ideals of A.

The following definition is due to J.A. Erdos [11].

4.2.2 DEFINITION : Let us define the representation ρ_e of A into the algebra of bounded linear operators on H_e by

$\rho_e(a)x = ax,\ x \in H_e.$

It is easiby seen that ρ_e is a representation because $H_e = Ae$ is a left ideal. Because $y^*ax = (a^*y)^* x$, it follows that ρ_e is adjoint preserving.

Regarding single elements of a C^*-algebra A and the representation defined above the following result has been proved by J.A. Erdos [11].

4.2.3 <u>THEOREM</u> : If s is a single element of A then $\rho_e(s)$ has rank one or zero. Every rank one operator on H_e is the image under ρ_e of some single element s of A. Also if $s \in \sigma_e$ then $||\rho_e(s)|| = ||s||$.

<u>PROOF</u> : The range of $\rho_e(s)$ is sAe and if s is single then sAe is zero or one dimensional by Theorem 3.2.6. Hence $\rho_e(s)$ has rank one or zero. Let p and q be non-zero vectors of H_e. By the definition of inner product on $Ae = H_e$ (Theorem 3.2.9), for $x \in H_e$, we have $p^*x = \langle x, p \rangle e$ and since $q = qe$, $qp^* \in \sigma_e$ and we have

$$\rho_e(qp^*)x = qp^*x = \langle x, p \rangle q.$$

Hence $\rho_e(qp^*) = p \otimes q$ and every rank one operator on H_e is of this form.

It is to be noted that if aeb is a non-zero element of σ_e than $aebb^*e \neq o$ as e is single and hence $\rho_e(aeb) \neq o$. Now for any non-zero element s of σ_e, we have

$$||\rho_e(s)||^2 = ||\rho_e(s)^* \rho_e(s)|| = ||\rho_e(s^*s)||$$

$$= \sup \{||\rho_e(s^*s)x|| : x \in H_e, ||x|| \leq 1\}$$

$$= \sup \{||s^*sx|| : x \in A_e, ||x|| \leq 1\}.$$

Using Lemma 3.2.3 we have $s^*s = kf$ where, as $s \neq o$, k is a non-zero complex number and f is a non-zero single self-adjoint idempotent. Since

$$||f|| = ||f^2|| = ||ff^*|| = ||f||^2$$

implies that $||f|| = 1$ and hence on taking norms

$$||s^*s|| = ||s||^2 = ||kf|| = |k|.$$

Thus

$$||\rho_e(s)||^2 = \sup \{|k|.||fx|| : x \in A_e, ||x|| \leq 1\}$$

Because $f \in A_e$, there exists $y \in A_e$ such that $fy \neq o$. Taking $x = fy/||fy||$ we get $||\rho_e(s)||^2 \geq |k| = ||s||^2$. But clearly $||\rho_e(s)|| \leq ||s||$ and hence for all $s \in \sigma_e$, we get $||\rho_e(s)|| = ||s||$.///

The following definition will be used in the sequel.

4.2.4 <u>DEFINITION</u> : A non-zero representation ρ of A on a Hilbert space H is said to be irreducible (topologically) if for all non-zero h in H, $\{\rho(a)h : a \in A\}$ is dense in H.

4.2.5 <u>COROLLARY</u> : ρ_e is irreducible.

4.2.6 THEOREM : [11] If ρ is a non-zero continuous irreducible representation of A on a Hilbert space H which is not necessarily adjoint preserving and for some single element s, $\rho(s) \neq o$ then ρ is similar to ρ_e where e is the single self-adjoint idempotent such that s = se. If ρ is adjoint preserving then ρ is unitarily equivalent to ρ_e.

PROOF : Since $\rho(s) = \rho(se) = \rho(s)\,\rho(e) \neq o$, $\rho(e)$ is a non-zero idempotent on H. Hence there exists a vector h in H such that $||h|| = 1$ and $(e)\,h = h$. Let us define the operator T from H_e to H by

$$Tx = \rho(x)\,h,\ x \in H_e.$$

Because ρ is continuous, there exists a positive constant k such that $||\rho(a)|| \leq k||a||$. Hence $||Tx|| \leq k\,||x||$. Also as $x = xe$, $x^*x = \lambda e$ for some constant λ and $||x^*x|| = |\lambda|$. Then $||\rho(x^*)||\ ||\rho(x)h\ || \geq ||\rho(x^*x)h|| = ||\lambda h|| = |\lambda| = ||x||^2$. Therefore since $||\rho(x^*)|| \leq k||x^*|| = k||x||$, $||Tx|| = ||\rho(x)h|| \geq k^{-1}\ ||x||$ for $x \in H_e$. Now as for all $a \in A$, $\rho(a)\ h = \rho(a)\,\rho(e)\ h = \rho(ae)h$, and since ρ is irreducible, the set $\{\rho(x)h : x \in H_e\}$ is a dense subset of H. From above, the operator mapping $\rho(x)h$ onto x is injective and bounded on this dense subset of H. Hence it may be extended by continuity to a bounded operator from H to H_e which is the inverse of T. Then for all $x \in H_e$, $T^{-1}\rho(a)\ Tx = T^{-1}\rho(ax)\ h = ax = \rho_e(a)x$ and thus ρ is similar to ρ_e.

If ρ is adjoint preserving it is automatically continuous and $||\rho(a)|| \leq ||a||$ for all a in A ([9], 1.3.7). Putting k=1 in above inequilities shows that in this case T is isometric and invertible and thus T is unitary, which completes the proof.///

4.2.7 <u>COROLLARY</u> : For any non-zero single element s of A there is one and only one unitary equivalence class $[\rho]$ of irreducible (adjoint preserving) representations such that $\rho(s) \neq o$ for $\rho \varepsilon [\rho]$. Also if $\rho \varepsilon [\rho]$ then $\rho(s)$ has rank one and $||\rho(s)|| = ||s||$.

Following is the main result of this section proved by J.A. Erdos.

4.2.8 <u>THEOREM</u> : There exists an isometric representation of the C^*-algebra A such that the image of each non-zero single element has rank one.

<u>PROOF</u> : Let $\{[\rho_\nu] : \nu \varepsilon \Gamma\}$ be the set of all unitary equivalence classes of irreducible representation of A, where Γ is an index set. Let us denote by $\{\rho_\nu : \nu \varepsilon \Gamma\}$ the set consisting of one representative from each equivalence class and let H_ν be the Hilbert space of ρ_ν. Now define a Hilbert space H and a representation π of A on H by

$$H = \oplus \{ H_\nu : \nu \varepsilon \Gamma \},$$
$$\pi(a) = \oplus \{ \rho_\nu(a) : \nu \varepsilon \Gamma \}.$$

Theorem 2.7.3 in [9] states that there exists a set $\{\rho_i : i \in I\}$ of irreducible representations of A such that $\|a\| = \sup\{\|\rho_i(a)\| : i \in I\}$. We have

$$\|\pi(a)\| = \sup\{\|\rho_\nu(a)\| : \nu \in \Gamma\} \geq \|a\|$$

because $\|\rho(a)\|$ depends only on the equivalence class of ρ.

Since the opposite inequility holds for all representations (see 1.3.7 of [9]), π is isometric.

Using Corollary 4.2.7, for any non-zero single element s of A, $\rho_\nu(s) \neq 0$ for exactly one ν and for this ν, $\rho_\nu(s)$ has rank one. Therefore, $\pi(s)$ has rank one./**//**/

The following result which guarantees the existence of an isometric *-representation of a C^*-algebra A on a Hilbert space H preserving compactness and finite-dimensionality of elements of A was proved by Ylinen [28]. More precisely we have the following :

4.2.9 <u>THEOREM</u> : Let A be a C^*-algebra. There exists an isometric *-representation π of A on a Hilbert space H such that $u \in A$ is a compact element of A if and only if $\pi(u)$ is a compact operator on H. Furthermore the linear operator $x \longrightarrow uxu$ on A has finite rank if and only if $\pi(u)$ has finite rank.

<u>PROOF</u> : Firstly, we show that there is an isometric *-representation π of A such that $\pi(u)$ is a compact

operator for each $u \in C$, and $\pi(u)$ has finite rank if $u \in F$. This is clear if zero is the only compact element of A. So let us assume that A has a non-zero compact element. Since A is a C^*-algebra, the socle of A coincides with F ([27], Theorem 5.1) and $\overline{F} = C$ ([27], Theorem 3.10). Hence it follows that the socle of A exists and coincides with F and its norm closure equals C. Therefore by virtue of Theorem 4.2.8 and Lemma 3.2.10, there exists an isometric $*$-representation π of A on a Hilbert space H such that $\pi(u)$ has finite rank for each $u \in F$, and $\pi(u)$ is a compact operator on H if $u \in C$.

Conversely, if $\pi(u)$ is a compact operator (resp. has finite rank), it is a compact element of $\pi(A)$ (resp. the operator $T \longrightarrow \pi(u) T \pi(u)$ on $\pi(A)$ has finite rank) using Theorem 2.3.2 and hence $u \in C$ (resp. $u \in F$). ///

4.3.1 In this section it will be shown that there exists an $*$-isometry of a primitive B^*-algebra A into the algebra of bounded linear operators on some Hilbert space H such that C, the set of all compact elements of A is mapped onto the algebra of compact operators on H.

In the sequel, for a given Banach space X the algebra of bounded linear operators on X will be denoted by L(X) and the subalgebra of compact operators on X by K (X).

Before proving the above mentioned result we need the following :

4.3.2 <u>LEMMA</u> : [2] Let A be a semi-simple Banach algebra such that $C \neq o$. Then A contains minimal right (left) ideals.

<u>PROOF</u> : If $\{x_n\} \subset C$ is a sequence and $x_n \longrightarrow x$ then the sequence of compact operators $a \longrightarrow x_n a x_n$ converges in the operator norm to the operator $a \longrightarrow xax$. Hence the latter is compact and thus $x \in C$ that is C is closed. Now let $o \neq x \in C$, then $xA \subset C$. This is so because the map $a \longrightarrow xyaxy$ is the composition of the maps $a \longrightarrow ya$, $a \longrightarrow xax$ and $a \longrightarrow ay$ and this is compact as $a \longrightarrow xax$ is so. If J is the closure of xA then $J \subset C$ as C is closed. Hence J is a compact Banach algebra. Also, J is not a radical algebra since A is semi-simple.. Theorem 4.3 in [1] states that if A is a compact Banach algebra which is not a radical algebra then A contains a non-zero idempotent e such that eAe is finite-dimensional, we get that J contains an idempotent e such that eJe is finite-dimensional. But $e \in J$ implies that $eA \subset J$, hence $eAe = e(eA)e \subset eJe$. Therefore eAe is finite-dimensional, which implies that e belongs to the socle of A using Theorem 7.2 of [1] which says that for a non-zero element x of a semi-simple Banach algebra A, the operator $a \longrightarrow xax$ $(a \in A)$ has finite rank if and only if the socle of A exists and contains x. Hence the socle of A is non-zero and hence A has minimal right and left ideals.///

4.3.3 **THEOREM** : [2] Let A be a primitive B^*-algebra such that $C \neq o$. Then there exists an $*$-isometry of A into the algebra $L(H)$ for some Hilbert space H, such that C is mapped onto $K(H)$.

PROOF : By Lemma 4.3.2 and ([21], Lemma 4.10.1) which says that if in an arbitrary $*$-algebra $A, x^*x = o$ implies $x = o$, x in A, then every minimal left ideal ℓ in A is of the form $\ell = Ap$, where p is a unique hermitian idempotent. A similar result holds for right ideals. Therefore, a minimal projection p in A can be found. Let H be the Banach space Ap. Then if $x, y \in H$ we have $y^*x \in pAp$ which consists of scaler multiples of p by the Gel'fand-Mazur theorem. Let us define an inner product on H by $\langle x, y \rangle p = y^*x$ in the usual way. Moreover, H is a Hilbert space with respect to this inner product, and the norm of H is equivalent to the norm inherited from A using ([21], 4.10.6).

Now let us define π from A into $L(H)$ by $\pi(x)(a) = xa$ Then π is a continuous $*$-representation of A on H., and π is faithful since A is primitive. Also since a B^*-algebra has a unique norm with the B^*-property, it follows that π is an isometry.

If $x, y, a \in H$, then $\pi(xy^*)(a) = xy^*a = \langle a, y \rangle x = (x \otimes y)(a)$. Thus $\pi(A)$ has all the operators of finite rank on H. Since $K(H)$ is the closure of finite-rank operators, and π is an isometry, implies that $K(H) \subseteq \pi(A)$. If

$T \in K(H)$ then the map $S \longrightarrow TST$ is compact by Theorem 2.3.2. Therefore, the map $\pi(y) \longrightarrow T\,\pi(y)\,T$ is compact on $\pi(A)$. As $K(H) \subseteq \pi(A)$ which implies that $k = \pi(x)$ for some $x \in A$, and the map $\pi(y) \longrightarrow \pi(xyx)$ is compact on $\pi(A)$. Hence, the map $y \longrightarrow xyx$ which is the composition of $y \longrightarrow \pi(y) \longrightarrow \pi(xyx) \longrightarrow xyx$ is compact on A, implies that x is compact i.e. $x \in C$. Thus $K(H) \subseteq \pi(C)$.

Let us note that if $x,y \in H$, then

$$||x|| \sup_{||a|| \leq 1} |< a,y >| = |(x \otimes y)| = |\,\pi(xy^*)| =$$

$$||xy^*|| \leq ||x||\ ||y||,$$ where $|.|$ is the operator norm for $L(H)$. Hence if $y \neq o$ then the map $x \longrightarrow ||xy||$ is a norm on H which is equivalent to the original norm of H.

Now we show that $\pi(C) \subseteq K(H)$. To this end, let $x \in C$ and $\{a_n\}$ be a sequence in the unit ball of H. Let us choose $y \in H$ such that $yx \neq o$. Then $\{a_n y\}$ is a bounded sequence, and since $x \in C$, there exists a subsequence $\{a_{n_k}\}$ such that $\{x\, a_{n_k} yx\}$ converges in A. As $yx \neq o$, we get that $\{\,\pi(x)a_{n_k}\} = \{xa_{n_k}\}$ converges using the observation made prior to this paragraph. Thus, $\pi(x)$ is a compact operator. Hence $\pi(C) \subseteq K(H)$. ///

4.3.4 <u>COROLLARY</u> : If A is a primitive B^*-algebra then C, the set of compact elements of A, is an ideal and is equal to the closure of the socle of A.

In the context of C^*-algebras we have the following result proved by Astala and Ramanujan [3].

4.4.1 LEMMA : Let A be a C^*-algebra and $b \in A$.
Let us define $T_b : A \longrightarrow A$ such that $T_b(x) = bxb$, $x \in A$.
Then $b = o$ whenever rank $T_b = o$.

PROOF : Rank $T_b = o \Rightarrow T_b(b^*) = o \Rightarrow b^* T_b(b^*) = o \Rightarrow b^*bb^*b = o$.
But in a C^*-algebra $||b||^4 = ||b^*bb^*b||$ and hence the result follows.///

4.4.2 REMARK : The above lemma does not hold in general in Banach algebras.

4.4.3 DEFINITION : [3] In a Banach algebra B, for $b \in B$, a complex number λ is called an eigenvalue of b if there exists $a \in B$ such that $ba = \lambda a$. Such an element a is called a λ-eigenelement of b.

The multiplicity of the eigenvalue λ of b in B is defined as the cardinality of a maximal set of orthonormal, hermitian, 1-dimensional, λ-eigenelements of b.

The following result which is a modification of the results of Erdos [11] was proved by Astala and Ramanujan [3].

4.4.4 THEOREM : Let A be a C^*-algebra. Then there exists an isometric $*$-representation π of A on a Hilbert space H satisfying the conditions :

(a) Rank $\pi(b) = 1$ (in H) if and only if b is a 1-dimensional element in A ;

(b) A number $\lambda \in \mathbb{K}$ is an eigenvalue of $b \in A$ if and only if it is an eigenvalue of the operator $\pi(b)$. Moreover, the multiplicities are the same.

<u>PROOF</u> : Any isometric $*$-representation satisfies the only if part of (a) ; because, if rank $\pi(b) = 1$ then clearly, rank $T_b \leq 1$; but since $b \neq o$ we get that rank $T_b = 1$ i.e. b is 1-dimensional using Lemma 4.4.1.

Conversely, to find a representation π as in part (a), let us note first that by combining Theorem 4.2.8 and Lemma 3.2.10 we get the following :

There exists an isometric $*$-representation $\pi : A \longrightarrow L(H)$ such that $\pi(b)$ is an operator of rank one on H whenever Ab is a minimal left ideal in A.

Hence to prove part (a) it is sufficient to show that Ab is minimal for each 1-dimensional element b in A. Therefore assume b to be so. We claim that

$$(b^*b)^2 = \lambda(b^*b), \ \lambda > o \qquad (*)$$

It is to be noted that $a = b^*b$ is 1-dimensional and positive and $a^3 \neq o$, $a^6 \neq o$; therefore we get,

$$T_a(a^4) = \lambda^3 \, T_a \,(a), \ \lambda > o \ ; \text{ i.e. } a^6 = \lambda^3 a^3.$$

Thus $(*)$ is obtained on taking cube roots.

Now, let us define $f = \lambda^{-1} a = \lambda^{-1} (b^*b)$. Clearly

f is 1-dimensional, hermitian and idempotent ; Also,

$(b-bf)^*(b-bf) = b^*b - fb^*b - b^*bf + fb^*bf = o.$

Hence $b = bf$. Therefore $Ab = Abf \subset Af = Ab^*b \subset Ab$ and thus $Ab = Af$.

Finally, since in a C^*-algebra the square I^2 of any non-trivial ideal I does not vanish ([27], Lemma 5.1), the minimality of $Af = Ab$ follows from Corollary 2.1.9 in [21] which says that if e is a minimal idempotent in a Banach algebra B, then Be (resp. eB) is a minimal left (resp. right) ideal.

(b) We observe that the Hilbert space H constructed in Theorem 4.2.8 has the form

$$H = \bigoplus_{i \in I} Ae_i$$

where e_i, $ei \in I$, are 1-dimensional, hermitian, idempotent elements of A. By Theorem 3.2.9 each ideal Ae_i is a Hilbert space. Also, the representation π has the form

$$\pi = \bigoplus_{i \in I} \rho_i$$

where $\rho_i : A \longrightarrow L(Ae_i)$ is the representation

$$\rho_i(a)x = ax \ , \ x \in Ae_i$$

Furthermore, by the definition of inner product in Ae (see the definition preceding Theorem 3.2.9), two elements $h, g \in Ae$ are orthogonal if and only if $g^*h = o$.

Let us put

$H_\lambda = \{h \in H : \pi(b)\, h = \lambda h,\ h \neq o\}$

$A_\lambda = \{x \in A : bx = \lambda x,\ x \neq o\}.$

and let us take a $h \in H_\lambda$; then $h = (\ \ldots,\ h_i^{(i)},\ \ldots)$. Define $I_1 = \{i \in I : h_i \neq o\}$. Because $h_i = h_i e_i$ (e_i is idempotent) and since in the Erdos's construction (see Theorems 4.2.3, 4.2.6 and the definition of $*$ in Theorem 4.2.8) $e_i A e_j = o$ if $i \neq j$, we have

$$h_i^* h_j = o, \text{ if } i \neq j \qquad (* *)$$

Now let us define $f_i = h_i h_i^*$, $i \in I$. Since each h_i is an eigenelement of b, each f_i is also an eigenelement of b. Also, by $(* *)$, the f_i's are orthogonal. Clearly they are also 1-dimensional and hermitian, as $h_i \in Ae_i$ and e_i is 1-dimensional. Hence the normalized f_i's form a part of the basis (= maximal orthonormal set of hermitian 1-dimensional idempotents) of A_λ.

Let $h^i = (h_j^i)_{j \in I} \in H$ be such that $h_j^i = o$, $i \neq j$, $h_i^i = h_i$. If now one can find $g \in H_\lambda$ with $g \neq o$ and $\langle g, h^i \rangle = o$ for every $i \in I_1$, then

$$g_i^* h_i = o \text{ for every } i \in I, \qquad (* * *)$$

Now let us put $k_i = g_i g_i^*$, and normalize if $g_i \neq o$. Then

$g_i g_i^* h_j h_j^* = g_i(g_i^* h_j)h_j^* = o$ if $i \neq j$ and also $g_i g_i^* h_i h_i^* = o$ by (* * *).

Hence $k_i f_j = o$ for all i,j and the partial basis (f_i) in A_λ has been enlarged. Continuing this process we establish a 1-1 correspondence, i.e., $h_i \longrightarrow f_i$ between the basis in H_λ and part of a basis of A_λ.

On the other hand, an eigenvalue of b is an eigenvalue of $\pi(b)$ with multiplicity no greater than as an eigenvalue of $\pi(b)$. Thus the proof of (b) is complete.//

The following definition is due to Astala and Ramanujan [3].

4.4.5 DEFINITION : The representation π, whose existence has been shown in the preceding theorem, is called an Erdos representation.

4.4.6 REMARK : Let $b \in A$ be a compact element. Then b^*b is also compact and hence Theorem 4.4.4 and the spectral decomposition of Hermitian operators on a Hilbert space yields,

$$b^*b = \sum_{k=0}^{\infty} \lambda_k^2 e_k, \quad \lambda_0 \geq \lambda_1 \geq \ldots \geq o, \qquad \text{(I)}$$

where e_k's are orthonormal, self-adjoint and 1-dimensional elements of A. Consequently $|b| = (b^*b)^{1/2}$ is compact. We also have the following :

4.4.7 PROPOSITION : [3] Let $b \in A$ be compact and $|b|$, λ_k, e_k be as in (I). Then

$$b = \sum_{k=0}^{\infty} be_k \qquad \text{(II)}$$

where each element be_k, $k = 0,1,2 \ldots$, is 1-dimensional with $\|be_k\| = \lambda_k$.

PROOF : Let $\pi : A \longrightarrow L(H)$ be an Erdos-representation. Since $|\pi(b)| = (\pi(b)^*\pi(b))^{1/2} = (\pi(b^*)\pi\ (b))^{1/2} = (\pi(b^*b))^{1/2} = \pi(b^*b)^{1/2} = \pi(|b|)$ and $\pi(b)$ has its polar decomposition $\pi(b) = U|\pi(b)|$ we get,

$$\pi(b) = U\pi(|b|) = U\ (\sum_{k=0}^{\infty} \lambda_k\ e_k) = \Sigma\ U\pi(|b|)\ \pi\ (e_k)$$

$= \pi(\Sigma\ be_k)$.///

4.4.8 NOTE : If $b \in A$ is a finite-dimensional element, then $|b|$ is of the form $b = \sum_{k=1}^{n} \lambda_k e_k$ for some $n \in \mathbb{N}$. The rank or dimension of b is defined to be n, the number of non-zero eigenvalues of $|b|$, with multiplicities taken to into account. By Proposition 4.4.7 each n-dimensional element can be written as the sum of n linearly independent 1-dimensional elements. Also, if $\pi : A \longrightarrow L(H)$ is an Erdos representation then rank b = rank $\pi(b)$, the usual rank of the operator $\pi(b)$ in $L(H)$.

The following result connects the approximation numbers to that of eigenvalues of an element in a C^*-algebra.

4.4.9 <u>THEOREM</u> : [3] Let $b \in A$ be compact and $|b|$, (λ_n) and (e_n) as in (I). Then

$a_n(b) = \lambda_n$, where $a_n(b)$ is the n-th approximation number of b.

<u>PROOF</u> : Let $\pi : A \longrightarrow L(H)$ be an Erdos representation. From Proposition 4.4.7 we get $b = \sum_{k=o}^{\infty} be_k$, where be_k is 1-dimensional ; then $x = \sum_{k=o}^{n-1} be_k$ has rank n and so, if $\pi(b) = U\pi(|b|)$ is the polar representation of $\pi(b)$,

$$a_n(b) \leq || \sum_{k=n}^{\infty} be_k|| = || \sum_{k=n}^{\infty} U\pi(|b|)\pi(e_k)||$$

$$= || U \sum_{k=n}^{\infty} \lambda_k \; \pi(e_k)|| \leq || \sum_{k=n}^{\infty} \lambda_k \; \pi(e_k)||.$$

But since $\lambda_n \geq \lambda_{n+1} \geq \dots$ and since the e_k's are orthonormal, we get $a_n(b) \leq \lambda_n$.

To show the reverse inequility, we note that the operators $\pi(e_k)$ are of the form $\pi(e_k) = h_k \otimes h_k$, $h_k \in H$, $k \in \mathbb{N}$ where

$$(h \otimes h')\ell = \langle h', \ell \rangle h; \; h, h', \ell \in H.$$

Thus

$$\pi(b) = U \sum_{k=o}^{\infty} \lambda_k \; \pi(e_k) = U \sum_{k=o}^{\infty} \lambda_k h_k \otimes h_k$$

$$= \Sigma \; \lambda_k \; (U(h_k)) \otimes h_k.$$

Since the sets (h_k) and $(Uh_k) \subset H$ are orthonormal it follows that the distance of $\pi(b)$ to the rank-n operators in M is λ_n. Hence

$$||b-x|| = ||\pi(b) - \pi(x)|| \geq \lambda_n$$

for each n-dimensional element $x \in A$.

4.4.10 <u>COROLLARY</u> : If $\pi : A \longrightarrow L(H)$ is an Erdos representation and $b \in A$ is compact then

$$a_n(b) = a_n(|b|) = a_n(\pi(b)).$$

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